BRAID GROUPS, FREE GROUPS, AND THE LOOP SPACE OF THE 2-SPHERE

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ABSTRACT. The purpose of this article is to describe connections between the loop space of the 2-sphere, Artin's braid groups, a choice of simplicial group whose homotopy groups are given by modules called Lie(n), as well as work of Milnor [25], and Habegger-Lin [17, 22] on "homotopy string links". The current article exploits Lie algebras associated to Vassiliev invariants in work of T. Kohno [19, 20], and provides connections between these various topics.

Two consequences are as follows:
(1) the homotopy groups of spheres are identified as "natural" sub-quotients of free products of pure braid groups, and
(2) an axiomatization of certain simplicial groups arising from braid groups is shown to characterize the homotopy types of connected CW-complexes.

1. A TALE OF TWO GROUPS PLUS ONE MORE

In 1924 E. Artin [1, 2] defined the n-th braid group B_n together with the n-th pure braid group P_n, the kernel of the natural map of B_n to Σ_n the symmetric group on n-letters. It is the purpose of this article to derive additional connections of these groups to homotopy theory, as well as some overlaps with algebraic, and topological properties of braid groups.

This article gives certain relationships between free groups on n generators F_n, and braid groups which serve as a bridge between different structures. These connections, at the interface of homotopy groups of spheres, braids, knots, and links, and homotopy links, admit a common thread given by a simplicial group.

Recall that a simplicial group Γ_* is a collection of groups

$$\Gamma_0, \Gamma_1, \cdots, \Gamma_n, \cdots$$

together with face operations

$$d_i : \Gamma_n \to \Gamma_{n-1},$$

and degeneracy operations

$$s_i : \Gamma_n \to \Gamma_{n+1},$$

for $0 \le i \le n$. These homomorphisms are required to satisfy the standard simplicial identities.

One example is Milnor's free group construction $F[K]$ for a pointed simplicial set K with base-point $*$ in degree zero. The simplicial group $F[K]$ in degree n is the free group generated by the n simplices K_n modulo the single relation that $s_0^n(*) = 1$.

In case K is reduced, that is K consists of a single point in degree zero, the geometric realization of $F[K]$ is homotopy equivalent to $\Omega\Sigma|K|$ [25]. The first theorem below addresses one property concerning the simplicial group given by $F[\Delta[1]]$ where $\Delta[1]$ is

the simplicial 1-simplex, and a simplicial group given in terms of the pure braid groups described next.

A second example is given by the simplicial group which in degree n is given by $\Gamma_n = P_{n+1}$, the $(n+1)$-st pure braid group, and which is elucidated in [10, 3]. The face operations are given by deletion of a strand, while the degeneracies are gotten by "doubling" of a strand. This simplicial group is denoted AP_*.

Theorem 1.1. *The loop space (as simplicial groups) of the simplicial group* AP_*, $\Omega(\mathrm{AP}_*)$, *is isomorphic to* $F[\Delta[1]]$ *as a simplicial group.*

A connection between Artin's braid group and the loop space of the 2-sphere is given next using the feature that the second pure braid group is isomorphic to the integers with a choice of generator denoted $A_{1,2}$. Notice that the simplicial circle S^1 has a single non-degenerate point in degree 1 given by $\langle 0, 1 \rangle$. Thus there exists a unique map of simplicial groups

$$\Theta : F[S^1] \to \mathrm{AP}_*$$

such that $\Theta(\langle 0, 1 \rangle) = A_{1,2}$. One of the theorems stated in [10] is as follows.

Theorem 1.2. *The morphism of simplicial groups*

$$\Theta : F[S^1] \to \mathrm{AP}_*$$

is an embedding. Hence the homotopy groups of $F[S^1]$ *are natural sub-quotients of* AP_*, *and the geometric realization of quotient simplicial set* $\mathrm{AP}_*/F[S^1]$ *is homotopy equivalent to the* 2*-sphere. Furthermore, the image of* Θ *is the smallest simplicial subgroup of* AP_* *which contains* $A_{1,2}$.

This theorem gives that the homotopy groups of $F[S^1]$, those of the loop space of the 2-sphere, are given as "natural" sub-quotients of the braid groups, a result related to work of the second author [30]. The proof of the above theorem sketched in [10] relies heavily on the structure of a Lie algebra arising from the "infinitesimal braid relations" as Vassiliev invariants of braids by work of T. Kohno [19, 20, 21], Falk, and Randell [15], as well as work of V. Drinfel'd [13] on the KZ equations. The precise details in section 9 here depend heavily on the specific structure of this Lie algebra.

An example is listed next. The commutator of the braids x_1, and x_2 in the third braid group as listed below in section 9 represents the Hopf map $\eta : S^3 \to S^2$. The braid closure of this commutator gives the Borromean rings.

An analogue for all spheres arises at once by taking coproducts of simplicial groups $\mathrm{AP}_* \vee \mathrm{AP}_*$ which in degree n is given by the free product $P_{n+1} \amalg P_{n+1}$.

Corollary 1.3. *The smallest simplicial subgroup of* $\mathrm{AP}_* \vee \mathrm{AP}_*$ *which contains* $P_2 \amalg P_2$ *in degree* 1 *is isomorphic to* $F[S^1] \vee F[S^1]$. *Hence* ΩS^n *is a retract, up to homotopy, of the geometric realization of the simplicial subgroup* $F[S^1] \vee F[S^1]$ *for any* $n \geq 2$ *by the Hilton-Milnor theorem.*

In addition, the Lie algebraic methods used to prove this theorem suggest that the methods might be useful to study whether related maps are faithful. A "Lie algebraic/homological" criterion for testing whether a representation of P_n is faithful is given in [9].

The results above suggest an axiomatization of certain families of simplicial groups. One application of Theorems 1.1, and 1.2 is listed next. Let $\mathcal{B}$ denote the smallest full sub-category of the category of reduced simplicial groups which satisfies the following properties:

(1) The simplicial group AP_* is in $\mathcal{B}$.
(2) If Π, and Γ are in $\mathcal{B}$, then the coproduct $\Pi \vee \Gamma$ is in $\mathcal{B}$.
(3) If Π is in $\mathcal{B}$, and Γ is a simplicial subgroup of Π, then Γ is in $\mathcal{B}$.

(4) If Π is in $\mathcal{B}$, and Γ is a simplicial quotient of Π, then Γ is in $\mathcal{B}$.

To be precise, the authors are unaware of a specific reference for the definition of the object given by a simplicial subgroup. There are two natural, and equivalent definitions given in section 12.

Theorem 1.4. *Let $X(i)$, $i = 1, 2$ denote path-connected CW-complexes with a continuous function $f : X(1) \to X(2)$. Then there exist elements $\Gamma_{X(i)}$, together with a morphism $\gamma : \Gamma_{X(1)} \to \Gamma_{X(2)}$ in $\mathcal{B}$ such that the loop space $\Omega(X(i))$ is homotopy equivalent to the geometric realization of $\Gamma_{X(i)}$, and the induced map $|\gamma| : |\Gamma_{X(1)}| \to |\Gamma_{X(2)}|$ is homotopic to $\Omega(f)$.*

One property concerning these connections is that the Lie algebra obtained from the descending central series of a discrete group has several disparate features here. One is that the structure of the resulting Lie algebra gives the method for proving that map in the proof of Theorem 1.2 is an injection. The second feature is that this Lie algebra is used to give the axiomatization given in Theorem 1.4. As remarked above, this Lie algebra is used to characterize Vassiliev invariants of pure braids [19, 20]. Lastly, this Lie algebra as well as the Lie algebra obtained from the mod-p descending central series for a discrete group gives the classical the Bousfield-Kan spectral sequence, as well as the classical unstable Adams spectral sequence for which the descending central series is replaced by the mod-p descending central series.

It is natural to consider other quotients of the simplicial groups, and Lie algebras here such as those arising in work of Habegger-Lin, [17]. That is the purpose of the next few remarks with one goal given by a construction of natural quotients of the pure braid groups in Proposition 3.4, as well as 2.6.

A naturality argument provides a bridge between the above theorems, certain quotients of the braid groups as well as the modules Lie(n) related to the cohomology of the pure braid groups. To make this naturality argument precise, a setting to provide natural quotients of the braid group is required. This setting provides an identification of a classical representation of the pure braid group in the automorphism group of a free group due to E. Artin [1, 2] in terms of a "universal" semi-direct product, the so-called holomorph of a group. The utility of this information is that it provides a direct method for constructing natural quotients of the braid groups which fit within the contexts of homotopy theory, links, and "homotopy string links", as well as intermediate analogues. In addition, features of the holomorph give a direct proof of Artin's classic result that his representation of the braid group in the automorphism of a free group is faithful. This proof then extends at once to other natural quotients. An extension of this information is applied to certain choices of simplicial groups one of which arises from "homotopy string links".

Related results are given in [10, 3].

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2. On the holomorph of a group, and the pure braid groups as group extensions

The purpose of this section is to give methods for direct constructions of quotients of the pure braid groups via natural universal split group extensions. One application below is to extend work above on $\Theta_n : F_n \to P_{n+1}$ of Theorem 1.2 to the setting of homotopy string links.

Consider a discrete group π together with the universal semi-direct product $\mathrm{Hol}(\pi)$, as elucidated in work of M. Voloshina [29] with some details given below. The group $\mathrm{Hol}(\pi)$ is "the natural" split extension of $\mathrm{Aut}(\pi)$ by π,

$$1 \to \pi \to \mathrm{Hol}(\pi) \to \mathrm{Aut}(\pi) \to 1$$

where $\mathrm{Aut}(\pi)$ is the automorphism group of π. More precisely, the group $\mathrm{Hol}(\pi)$, as a set, is a product $\mathrm{Aut}(\pi) \times \pi$, but the group structure is defined by the product

$$(f, x) \cdot (g, y) = (f \cdot g, g^{-1}(x) \cdot y)$$

for f, g in $\mathrm{Aut}(\pi)$, and x, y in π. Hence,

$$(f, 1)^{-1} \cdot (1, y) \cdot (f, 1) = (1, f^{-1}(y)).$$

The topological analogue of this construction is the universal bundle with section having fibre a given $K(\pi, 1)$:

$$E\mathrm{Aut}(\pi) \times_\pi K(\pi, 1) \to B\mathrm{Aut}(\pi)$$

where $\mathrm{Aut}(\pi)$ acts by "left translation" on π. This construction will be used below to describe groups analogous to the braid groups obtained from a classical representation due to Artin.

Let F_n denote the free group on n-letters with generators $x_1, x_2, \cdots, x_n$. Artin gave a homomorphism from the n-th braid group to $\mathrm{Aut}(F_n)$ [2, 1, 4, 24]

$$A : B_n \to \mathrm{Aut}(F_n)$$

which is described below. One consequence is that the restriction of A to P_n identifies Artin's representation of P_{n+1} inductively arising from a pull-back in Corollary 2.6 of the two maps $A : P_n \to \mathrm{Aut}(F_n)$, and $\mathrm{Hol}(F_n) \to \mathrm{Aut}(F_n)$.

To describe this structure, recall the following results due to E. Artin [1, 2, 4, 24] where the commutator is defined by the equation $[a, b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$. The braid group on n-strands B_n is generated by elements, σ_i for $1 \leq i \leq n-1$ with the well-known relations

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \geq 2$, and
(2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all i.

Artin's representation of B_n is stated in the next theorem.

Theorem 2.1. *There is a homomorhism*

$$A : B_n \to \ \mathrm{Aut}(F_n)$$

obtained by setting

    (1) $A(\sigma_s)(x_j) = x_j$ *if* $j \neq s, s+1$,
    (2) $A(\sigma_s)(x_s) = x_{s+1}$, *and*
    (3) $A(\sigma_s)(x_{s+1}) = x_{s+1}^{-1} x_s x_{s+1} = x_s[x_s, x_{s+1}]$.

Thus

    (1) $A(\sigma_s)^{-1}(x_j) = x_j$ *if* $j \neq s, s+1$,
    (2) $A(\sigma_s)^{-1}(x_s) = x_s x_{s+1} x_s^{-1}$, *and*
    (3) $A(\sigma_s)^{-1}(x_{s+1}) = x_s$.

The pure braid group P_n is generated by elements $A_{r,s}$ for $1 \leq r < s \leq n$. The element $A_{r,s}$ may be thought of as linking the s-th strand around the r-th strand for $1 \leq r < s \leq n$ with a fixed orientation together with an explicit formula given by

$$A_{r,s} = \alpha(r,s) \cdot \sigma_r^2 \cdot \alpha(r,s)^{-1}$$

where

$$\alpha(r,s) = \sigma_{s-1} \cdot \sigma_{s-2} \cdot \sigma_{s-3} \cdots \sigma_{r+1}.$$

Artin gave a complete set of relations for the pure braid group P_n [2, 1], or [24]. A restatement of Artin's relations in terms of commutators is listed next.

Theorem 2.2. *The group P_n is generated by elements*

$$A_{r,s}$$

for $1 \leq r < s \leq n$. A complete set of relations is given as follows:

    (1) *If either* $s < i$, *or* $k < r$, $A_{r,s} A_{i,k} A_{r,s}^{-1} = A_{i,k}$.
    (2) *If* $i < k < s$, $A_{k,s} A_{i,k} A_{k,s}^{-1} = A_{i,s}^{-1} A_{i,k} A_{i,s}$.
    (3) *If* $i < r < k$, $A_{r,s} A_{i,k} A_{r,s}^{-1} = A_{i,k}^{-1} A_{i,r}^{-1} A_{i,k} A_{i,r} A_{i,k}$.
    (4) *If* $i < r < k < s$, $A_{r,s} A_{i,k} A_{r,s}^{-1} = A_{i,s}^{-1} A_{i,r}^{-1} A_{i,s} A_{i,r} A_{i,k} A_{i,r}^{-1} A_{i,s}^{-1} A_{i,r} A_{i,s}$.

Furthermore, these relations are equivalent to the following relations.

    (1) *If either* $s < i$, *or* $k < r$, *then* $[A_{i,k} A_{r,s}] = 1$.
    (2) *If* $i < k < s$, *then* $[A_{i,k}, A_{k,s}^{-1}] = [A_{i,k}, A_{i,s}]$.
    (3) *If* $i < r < k$, $[A_{r,k}^{-1}, A_{i,k}^{-1}] = [A_{i,k}, A_{i,r}]$.
    (4) *If* $i < r < k < s$, $[A_{i,k}, A_{r,s}^{-1}] = [A_{i,k}, [A_{i,r}, A_{i,s}]]$.

The well-known values of $A : B_n \to \ \mathrm{Aut}(F_n)$ restricted to P_n given explicitly in [24] are listed next together with some additional related information.

Theorem 2.3. *The homomorphism $A : B_n \to \ \mathrm{Aut}(F_n)$ restricted to P_n is specified by the following formula:*

$$A(A_{i,k})(x_r) = \begin{cases} x_r & \text{if } r < i \text{ or } k < r, \\ x_i \cdot [x_i, x_k] & \text{if } r = i, \\ [x_k, x_i] \cdot x_k & \text{if } r = k, \text{ and} \\ x_r \cdot [x_r, [x_i, x_k]] & \text{if } i < r < k. \end{cases}$$

In addition, if $1 \leq j \leq n$, then

$$\sigma_i^{-1} A_{j,n+1} \sigma_i = A(\sigma_i^{-1})(A_{j,n+1}).$$

The action of P_n given by conjugating the element $A_{j,n+1}$ for $1 \leq j \leq n$ by $A_{r,s}$, $1 \leq r < s \leq n$ is given as follows:

$$A_{r,s}^{-1} A_{j,n+1} A_{r,s} = A(A_{r,s}^{-1})(A_{j,n+1})$$

This theorem specifies the action of the image of B_n in $\mathrm{Aut}(F_{n+1})$ on the set $A_{1,n+1}, A_{2,n+1}, \cdots, A_{n,n+1}$. It will be checked below that this action is precisely that given in the holomorph of a free group. The utility of this observation is that it then provides easy methods for constructing certain quotients of braid groups as follows.

Let $\mathbb{F}$ denote a free group with a choice of generator given by x. Let $G * H$ denote the free product of two groups G, and H. There are homomorphisms

$$e : \mathrm{Aut}(G) \to \mathrm{Aut}(G * \mathbb{F}),$$

and

$$\chi : G \to \mathrm{Aut}(G * \mathbb{F})$$

defined as follows.

Definition 2.4. The homomorphism $e : \mathrm{Aut}(G) \to \mathrm{Aut}(G * \mathbb{F})$ is defined on an element f of $\mathrm{Aut}(G)$ by the formula

$$[e(f)](z) = \begin{cases} f(z) & \text{if } z \text{ is in } G, \text{ and} \\ z & \text{if } z \text{ is in } \mathbb{F}. \end{cases}$$

The homomorphism $\chi : G \to \mathrm{Aut}(G * \mathbb{F})$ is defined on an element h in G by the formula

$$[\chi(h)](z) = \begin{cases} z & \text{if } z \text{ is in } G, \text{ and} \\ h \cdot z \cdot h^{-1} & \text{if } z \text{ is in } \mathbb{F}. \end{cases}$$

There is an induced function

$$E : \mathrm{Hol}(G) \to \mathrm{Aut}(G * \mathbb{F})$$

given by the formula

$$E(f, h) = e(f) \cdot \chi(h)$$

for f in $\mathrm{Aut}(G)$, and h in G.

The next classical proposition follows directly with the verification given below.

Lemma 2.5. *The following properties hold in* $\mathrm{Aut}(G * \mathbb{F})$:
  (1) $\chi(h) \cdot e(f) = e(f) \cdot \chi(f^{-1}(h))$,
  (2) *the function* $E : \mathrm{Hol}(G) \to \mathrm{Aut}(G * \mathbb{F})$ *is a well-defined homomorphism which restricts to e on* $\mathrm{Aut}(G)$, *and χ on G such that*
$$e(f^{-1}) \cdot \chi(h) \cdot e(f) = \chi(f^{-1}(h)),$$
  and
  (3) *the induced homomorphism*
$$E : \mathrm{Hol}(G) \to \mathrm{Aut}(G * \mathbb{F})$$
  is a monomorphism.

Proof. Notice that

$$\chi(h) \cdot e(f)(z) = \begin{cases} f(z) & \text{if } z \text{ is in } G, \text{ and} \\ h \cdot z \cdot h^{-1} & \text{if } z \text{ is in } \mathbb{F}. \end{cases}$$

$$e(f) \cdot \chi(f^{-1}(h))(z) = \begin{cases} f(z) & \text{if } z \text{ is in } G, \text{ and} \\ h \cdot z \cdot h^{-1} & \text{if } z \text{ is in } \mathbb{F}. \end{cases}$$

This relation is exactly that required in the multiplication for $\mathrm{Hol}(G)$ given by $(f, 1) \cdot (1, y) \cdot (f, 1)^{-1} = (1, f^{-1}(y))$, and thus E is a homomorphism which satisfies the property

$$e(f^{-1}) \cdot \chi(h) \cdot e(f) = \chi(f^{-1}(h)).$$

To finish, it suffices to check that $E : \mathrm{Hol}(G) \to \mathrm{Aut}(G * \mathbb{F})$ is a monomorphism. Notice that the kernel of E projects trivially to $\mathrm{Aut}(G)$, and thus the kernel is contained in G. But E restricted to G is a monomorphism, and the lemma follows.   □

The above constructions now give a direct check that Artin's representation of the pure braid group is obtained as natural pull-back. In addition, this identification gives a direct way to reproduce Artin's result well-known [4] that his representation is faithful.

Corollary 2.6. *The group* P_{n+1} *is isomorphic to the group* ζ_{n+1} *given as the pull-back defined by the following cartesian diagram:*

$$
\begin{array}{ccc}
\zeta_{n+1} & \xrightarrow{\ i\ } & P_n \\
\downarrow{\scriptstyle I} & & \downarrow{\scriptstyle A|_{P_n}} \\
\mathrm{Hol}(F_n) & \xrightarrow{\ j\ } & \mathrm{Aut}(F_n).
\end{array}
$$

Furthermore, Artin's representation of P_{n+1} *is given by the composite*

$$
P_{n+1} = \zeta_{n+1} \xrightarrow{\ I\ } \mathrm{Hol}(F_n) \xrightarrow{\ E\ } \mathrm{Aut}(F_{n+1}).
$$

Thus Artin's representation is faithful.

Proof. Let $x_i = A_{i,n+1}$. With this identification, the action of P_{n+1} on the free group F_n as specified in the extension given by the holomorph by the formulas

$$
(f,1)^{-1} \cdot (1,y) \cdot (f,1) = (1, f^{-1}(y)),
$$

and if $1 \leq j \leq n$. That is

$$
\sigma_i^{-1} A_{j,n+1} \sigma_i = A(\sigma_i^{-1})(A_{j,n+1})
$$

by Theorem 2.3. The next formula follows at once for $1 \leq r < s \leq n$:

$$
A_{r,s}^{-1} A_{j,n+1} A_{r,s} = A(A_{r,s}^{-1})(A_{j,n+1}) = A(A_{r,s}^{-1})(x_j).
$$

Thus these formulas agree with the action given by conjugation in P_{n+1} by $A_{r,s}$ for $1 \leq r < s \leq n$, and so the pull-back ζ_{n+1} is isomorphic to P_{n+1}.

The next step is to check that Artin's representation $A : P_n \to \mathrm{Aut}(F_n)$ is faithful, a statement which is trivially correct in case $n = 1$. The proof that A is faithful when restricted to P_n then follows by induction on n. That is, the pull-back of a monomorphism is again a monomorphism while P_{n+1} is the pull-back of a faithful representation given by $P_{n+1} \to \mathrm{Hol}(F_n)$ as in 2.6 followed by the injection $E : \mathrm{Hol}(F_n) \to \mathrm{Aut}(F_{n+1})$ 2.5.

To finish the proof that A is faithful when restricted to B_n, consider the commutative diagram

$$
\begin{array}{ccc}
B_n & \xrightarrow{\ A\ } & \mathrm{Aut}(F_n) \\
\downarrow & & \downarrow \\
\Sigma_n & \xrightarrow{\ \rho\ } & \mathrm{GL}(n, \mathbb{Z})
\end{array}
$$

for which $\rho : \Sigma_n \to \mathrm{GL}(n, \mathbb{Z})$ is the natural faithful representation. Thus any element in the kernel of $B_n \to \mathrm{Aut}(F_n)$ must be an element of P_n and is the identity by the previous remarks. The result follows.   □

3. Quotients of the pure braid group, and the maps $B_n \to \mathrm{Aut}(K_n)$

Consider the "reduced free group" due to Milnor[25, 26] used to analyze "homotopy string links", and studied in Habegger-Lin [17], [22]. The "reduced free group" K_n is defined as the quotient of F_n modulo the relations

$$[x_i, gx_ig^{-1}] = 1$$

where x_i is any generator of F_n, g is any element in F_n, and $[x,y]$ denotes the commutator $x \cdot y \cdot x^{-1} \cdot y^{-1}$. Define Λ_n to be the smallest normal subgroup of F_n containing all of the elements $[x_i, gx_ig^{-1}]$. Thus

$$K_n = F_n/\Lambda_n.$$

This group was rediscovered in a different context [6] to analyze Barratt's finite exponent conjecture where K_n is the quotient of the free group on n generators modulo the smallest normal subgroup containing the simple commutators where at least one generator appears twice. More precisely, let $\bar{\Lambda}_n$ denote the smallest normal subgroup of F_n containing all commutators of the form $[\cdots[x_{i_1}, x_{i_2}]x_{i_3}], \cdots]x_{i_t}]$ where $x_{i_j} = x_{i_k}$ for some $j < k$. The reduced free group is the quotient

$$K_n = F_n/\bar{\Lambda}_n.$$

The structure of this group as well as certain subgroups was analyzed in [6] by considering the group of units in a non-abelian version of an exterior algebra.

The next result gives that the action of B_n in $\mathrm{Aut}(F_n)$ descends to an action on K_n, although the action of the entire automorphism group $\mathrm{Aut}(F_n)$ does not descend to K_n.

Proposition 3.1. *The action of the group B_n regarded as subgroup of $\mathrm{Aut}(F_n)$ acting naturally on F_n descends to an action of B_n on K_n denoted*

$$\alpha : B_n \to \mathrm{Aut}(K_n)$$

and defined by setting

(1) $\alpha(\sigma_s)(x_j) = x_j$ *if* $j \neq s, s+1$,
(2) $\alpha(\sigma_s)(x_s) = x_{s+1}$, *and*
(3) $\alpha(\sigma_s)(x_{s+1}) = x_s[x_s, x_{s+1}] = x_{s+1}^{-1}x_sx_{s+1}$.

The next technical remark both follows by inspection, and gives an immediate proof of Proposition 3.1.

Lemma 3.2. *The following identity is satisfied in any group:*

$$[x^{-1}yx, vyv^{-1}] = x^{-1}[y, (xv)y(xv)^{-1}]x$$

A direct check of this identity is listed next for completeness.

$$[x^{-1}yx, vyv^{-1}] = x^{-1}\{(y^{-1}(xv)y^{-1}(v^{-1}x^{-1})\}y\{(xv)y(v^{-1})x^{-1}\}x$$

which equals $x^{-1}[y, (xv)y(xv)^{-1}]x$. This lemma is used to prove Proposition 3.1, a restatement of 3.4.

The proof of Proposition 3.1 is given next.

Proof. It suffices to see that the action B_n on F_n acting naturally through Artin's representation descends to an action on K_n. To check this point, notice that the relations in K_n are given by

$$[x_s, gx_sg^{-1}]$$

for all $1 \leq s \leq n$ and all g in F_n.

It must be checked that the relations are preserved for the cases in which

(1) x_s is replaced by x_{s+1}, and

(2) x_{s+1} is replaced by $x_{s+1}^{-1}x_sx_{s+1}$ with g an arbitrary element of F_n.

The case 1 is clear, and the case 2 is checked next.

Write $x_{s+1} = x$, and $x_s = y$. Then consider $[x_{s+1}, gx_{s+1}g^{-1}] = [x, gxg^{-1}]$. Replace x by $x^{-1}yx$ to obtain

$$[x^{-1}yx, (gx^{-1})y(xg^{-1})].$$

Use the identity in 3.2 $[x^{-1}yx, vyv^{-1}] = x^{-1}[y, (xv)y(xv)^{-1}]x$ to obtain the relation

$$[x^{-1}yx, (gx^{-1})y(xg^{-1})] = 1.$$

Thus Proposition 3.1 follows.   □

With these constructions, there are natural quotients of P_n, as well as an analogous simplicial group obtained from "homotopy string links". The reduced braid groups as well as reduced pure braid groups are defined next.

Definition 3.3. Proposition 3.1 gives a representation $\alpha : B_n \to \mathrm{Aut}(K_n)$. Define the reduced braid group rB_n as the image of α, and reduced pure braid groups rP_n as the image of α restricted to P_n.

Proposition 3.4. *There is a morphism of group extensions*

$$
\begin{array}{ccccc}
F_n & \xrightarrow{\;i\;} & P_{n+1} & \xrightarrow{\;d_i\;} & P_n \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\
K_n & \xrightarrow{\;i\;} & rP_{n+1} & \xrightarrow{\;rd_i\;} & rP_n
\end{array}
$$

where d_i denotes deletion of the $(i+1)$-st strand with rd_i the induced homomorphism. In addition, there are morphisms of groups

$$
\begin{array}{ccc}
P_n & \xrightarrow{\;s_j\;} & P_{n+1} \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
rP_n & \xrightarrow{\;rs_j\;} & rP_{n+1}
\end{array}
$$

where s_j denotes the degneracy given by "doubling" of the $(j+1)$-st strand with rs_j the induced homomorphism.

Thus there are quotients of P_n obtained by replacing F_q by K_q for all $1 \leq q \leq n-1$ via Definition 3.3. The resulting quotients give groups rP_n of Habegger-Lin, and Milnor which arises from the notion of link homotopy as described crudely in the next paragraph.

Consider the space of ordered n-tuples of smooth maps of S^1 in $\mathbb{R}^3$ having disjoint images. Milnor defines an equivalence relation of "link homotopy" by allowing strands from the same component of a link to pass over each other [25, 26]. Lin describes the infinitesimal link-homotopy relations on page 5 of [22].

For example, consider the trivial n-component link $\mathcal{L}_n$ in $\mathbb{R}^3$. Thus the fundamental group of the complement $\mathbb{R}^3 - \mathcal{L}_n$ is isomorphic to F_n by some choice of isomorphism. Let β denote a simple closed curve in the complement which represents an element b in F_n. Then both Milnor, and Habegger-Lin prove that the link $\mathcal{L}_n \cup \beta$ is link homotopically trivial if and only if the element b projects to the identity in K_n. [25, 26, 17, 22].

Theorem 3.5. *The natural quotient map $\rho : F_n \to K_n$ prolongs to a morphism of simplicial groups*

$$\rho : F[S^1] \to K[S^1]$$

for which $K[S^1]$ in degree n is K_n. The kernel of ρ, $\Gamma[S^1]$, is a free group in each degree.

(1) *The map ρ is a surjection of simplicial groups, and is thus a fibration with fibre given by a simplicial group which is free in each degree.*
(2) *The n-th homotopy group of $K[S^1]$, $\pi_n K[S^1]$, is isomorphic to $\oplus_{(n-1)!}\mathbb{Z}$, the $(n-1)$-st homology group of P_n given by $\mathrm{Lie}(n)$.*
(3) *The induced map $\pi_*(\rho) : \pi_*(F[S^1]) \to \pi_*(K[S^1])$ is an isomorphism in dimension 1, and 2. This map is zero in degrees greater than 2.*
(4) *If $n > 2$, here is a short exact sequence of homotopy groups*

$$0 \to \ \mathrm{Lie}(n+1) \to \ \pi_n\Gamma[S^1] \to \ \pi_n(F[S^1]) \to \ 0.$$

Proof. The quotient maps $\rho : F_n \to \ K_n$ commute with the face and degeneracies by 3.4. That ρ is a surjection of simplicial groups is a consequence of 3.4. In addition, the homotopy groups of $K[S^1]$ were determined in [6]. The theorem follows at once from standard properties of simplicial groups applied to $K[S^1]$ [6, 30].   □

The above remarks provide a comparison of the group theory associated to link homotopy to the group theory associated to the homotopy groups of the 2-sphere. In addition, there is a functor from simplicial groups to pro-groups which when specialized to $K[S^1]$ gives a group H_∞ [7] which is filtered with associated graded given by

$$E_0^*(H_\infty) = \oplus_{n \geq 1} Lie(n).$$

The framework above provides other natural quotients of F_n, as well as P_n which will be addressed elsewhere.

4. On link homotopy and related homotopy groups

The purpose of this section is to describe some connections of free groups, braid groups, homotopy groups of the 2-sphere. The subsequent sections will illustrate one connection between "link homotopy", Artin's pure braid groups, and the homotopy groups of certain choices of simplicial groups.

There is a braid invariant obtained from homotopy groups as follows. Consider the embedding $\Theta_n : F_n \to \ P_{n+1}$. Let C_n denote the chains in degree n for $F[S^1]$. That is the normal subgroup given by the intersection of the kernels of $d_1, d_2, \cdots, d_n$ (excluding d_0).Thus P_{n+1} is equal to a disjoint union of left cosets:

$$P_{n+1} = \mathrm{II}_{\alpha \epsilon S} x_\alpha C_n$$

where

$$\{x_\alpha | \alpha \epsilon S\}$$

is a complete set of distinct left coset representatives for C_n in P_{n+1}. Thus given a pure braid γ, assign the homotopy element given by the homotopy class of x where $\gamma = \alpha \cdot x$. As an example, this element is analyzed for three-stranded braids.

Example 4.1. Let $E_0^n(K_q)$ denote the n-stage of the descending central series for K_q modulo the $n + 1$-st stage. Then $E_0^n(K_3)$ has a basis as follows [7].

(1) If $n = 1$, a basis is x_1, x_2, and x_3.
(2) If $n = 2$, a basis is $[x_1, x_2]$, $[x_1, x_3]$, and $[x_2, x_3]$.
(3) If $n = 3$, a basis is $[[x_1, x_2], x_3]$, and $[[x_1, x_3], x_2]$.

A basis for $E_0^t(K_n)$ is $[[x_{i_1}, x_{i_{\tau(2)}}], x_{i_{\tau(3)}}], \cdots], x_{i_{\tau(t)}}]$ for all sequences $1 \leq i_1 < i_2 < i_3 < \cdots < i_t \leq n$ where τ runs over all elements in the symmetric group Σ_{t-1}.

A non-bounded cycle in F_2 is given by $[x_1, x_2]$. This cycle represents a choice of generator of $\pi_2 \Omega S^2 = \mathbb{Z}$ given by the classical Hopf map. In addition, the braid closure of $[x_1, x_2]$ gives the Borromean rings. The computations are direct, and left as an exercise.

5. On looping AP_*

The next theorem was proven in [10] while a detailed proof is included here for completeness.

Theorem 5.1. *The loop space (as simplicial groups) of the simplicial group AP_*, $\Omega(AP_*)$, is isomorphic to $F[\Delta[1]]$ as a simplicial group. Thus AP_* is contractible, and the realization of the simplicial set $AP_*/F[S^1]$ is homotopy equivalent to S^2.*

Remark 5.2. *The long exact homotopy sequence of the fundamental fibration sequence due to Fadell, and Neuwirth [14] can be regarded as determining the homotopy type of the loop space for the simplicial group obtained from the pure braid groups.*

Proof. John Moore gave a definition for the loop space $\Omega\Gamma_*$ of a reduced simplicial group Γ_* (a simplicial group for which Γ_0 consists of a single element) [28]. This procedure corresponds to the topological notion of looping in the sense that the loop space of the geometric realization of Γ_* is homotopy equivalent to the geometric realization of $\Omega\Gamma_*$. This process of looping a simplicial group is described next.

Define a simplicial group $E\Gamma_*$ where the group $E\Gamma_n$ in degree n is given by the group Γ_{n+1} with face, and degeneracies given by "shifting down by one". Then $\Omega\Gamma_*$, the looping of Γ_* in degree n, is defined to be the kernel of the map

$$d_0 : E\Gamma_n \to \Gamma_n$$

thus

$$\Omega\Gamma_n = \ker[d_0 : \Gamma_{n+1} \to \Gamma_n].$$

In the special case that Γ_n is P_{n+1}, then $E\Gamma_n$ is P_{n+2}. Recall that P_{n+1} is generated by symbols $A_{i,j}$ for $1 \leq i < j \leq n+1$. Furthermore, the map $d_0 : E\Gamma_n \to \Gamma_n$ is induced by the projection map $p_* : \pi_1(F(\mathbb{R}^2, n+2)) \to \pi_1(F(\mathbb{R}^2, n+1))$ where $p : F(\mathbb{R}^2, n+2) \to F(\mathbb{R}^2, n+1)$ is the map given by projection to the first $n+1$ coordinates.

The fibre of the map p is $\mathbb{R}^2 - Q_{n+1}$, and thus the kernel of p_* is isomorphic to F_{n+1}. It will be shown below that a choice of generators for this kernel, regarded as a simplicial set is $\Delta[1]$. Together with Theorem 1.2 giving that $\Theta : F[S^1] \to AP_*$ is an embedding, the result follows.

The kernel of the map $d_0 : P_{n+2} \to P_{n+1}$ which is given by deleting the last coordinate is given by the free group with generators $P_{n+2,j}$ for $1 \leq j < n+2$. A basis for the kernel is given by $\{P_{n+2,j} | 1 \leq j \leq n+1\}$.

Furthermore, the simplicial 1-simplex is given by $\langle 0^i, 1^{n+1-i} \rangle$ in degree n with $0 \leq i \leq n+1$ with the single relation that $\langle 0^{n+1} \rangle = 1$ in the simplicial group $F[\Delta[1]]$. There is a dimension-wise morphism of groups

$$\Psi : F[\Delta[1]] \to \Omega(AP_*)$$

defined by sending $\langle 0^i, 1^{n+1-i} \rangle$ to $P_{n+2,i}$ for $0 < i < n+2$. A direct check gives that this dimension-wise homomorphism is compatible with the face, and degeneracies. Since each map is an isomorphism, the map Ψ is an isomorphism of simplicial groups. The theorem follows. $\square$

6. On embeddings of residually nilpotent groups

Let $\rho : \pi \to G$ be a homomorphism between discrete groups. Let $\Gamma^n(\pi) = \Gamma^n$ denote the n-th stage of the descending central series for π. That is

(1) $\Gamma^1(\pi) = \pi$, and inductively
(2) $\Gamma^{n+1}(\pi) = [\Gamma^n, \pi]$, the subgroup generated by commutators $[\cdots [h_1, h_2] h_3] \cdots] h_q]$ with h_i in π, and $q \geq n+1$.

Define the associated graded

$$E_0^n(\pi) = \Gamma^n(\pi)/\Gamma^{n+1}(\pi),$$

and

$$E_0^*(\pi) = \oplus_{n \geq 1} E_0^n(\pi).$$

It is a classical and easily checked fact that the commutator

$$[-,-] : \pi \times \pi \to \pi$$

given by

$$[x, y] = x^{-1} \cdot y^{-1} \cdot x \cdot y$$

induces the structure of Lie algebra on $E_0^*(\pi)$. A group homomorphism $\rho : \pi \to G$ preserves the stages of the descending central series. Thus there is an induced morphism of associated graded Lie algebras

$$E_0^*(\rho) : E_0^*(\pi) \to E_0^*(G).$$

Definition 6.1. A discrete group Γ is said to be residually nilpotent group if

$$\bigcap_{i \geq 1} \Gamma^i(\pi) = \{\text{identity}\}$$

where $\Gamma^i(\pi)$ denotes the i-th stage of the descending central series for π.

Theorem 6.2. *Assume that π is a residually nilpotent group. Let*

$$\rho : \pi \to G$$

be a homomorphism of discrete groups such that the morphism of associated graded Lie algebras

$$E_0^*(\rho) : E_0^*(\pi) \to E_0^*(G)$$

is a monomorphism. Then ρ is a monomorphism.

Proof. Let x denote a non-identity element in the kernel of ρ. Since π is residually nilpotent, there exists a natural number n such that the element x is in $\Gamma^n(\pi)$ and not in $\Gamma^{n+1}(\pi)$. But then x projects to a non-identity element in $E_0^n(\pi)$, and thus has non-trivial image in $E_0^n(G)$ contradicting the fact that x is a non-identity element in the kernel of ρ. $\square$

If $F[S]$ is a free group generated by the set S, then $F[S]$ is residually nilpotent [24], and the next corollary follows at once.

Corollary 6.3. *If $F[S]$ is a free group generated by the set S, and*

$$E_0^*(\rho) : E_0^*(F[S]) \to E_0^*(G)$$

is a monomorphism of Lie algebras, then $\rho : F[S] \to G$ is a monomorphism of groups. If ρ is assumed to be an epimorphism, then it is an isomorphism.

Proof. If f induces an monomorphism on $E_0^*(F[S])$, then f is a monomorphism by the previous theorem. It suffices to note that f induces a surjection. The lemma follows. $\square$

This approach for testing whether a homomorphism ρ is faithful is suited for residually nilpotent groups π such as free groups, and pure braid groups P_{n+1}. A single case is addressed here in section 8 here. One, possibly useful case, is that of the Gassner representation.

7. The simplicial structure for AP_*

Recall that P_{n+1} is generated by symbols $A_{i,j}$ for $1 \leq i < j \leq n+1$. Artin's relations are listed in [24] as well as in section 3 here.

The face operations in the simplicial group AP_* are defined as follows:

$$d_t(A_{i,j}) = \begin{cases} A_{i-1,j-1} & \text{if } t+1 < i, \\ 1 & \text{if } t+1 = i, \\ A_{i,j-1} & \text{if } i < t+1 < j, \\ 1 & \text{if } t+1 = j, \\ A_{i,j} & \text{if } t+1 > i. \end{cases}$$

The degeneracy operations are defined as follows:

$$s_t(A_{i,j}) = \begin{cases} A_{i+1,j+1} & \text{if } t+1 < i, \\ A_{i,j+1} \cdot A_{i+1,j+1} & \text{if } t+1 = i, \\ A_{i,j+1} & \text{if } i < t+1 < j, \\ A_{i,j} \cdot A_{i,j+1} & \text{if } t+1 = j, \\ A_{i,j} & \text{if } t+1 > j. \end{cases}$$

That the face and degeneracies preserve the defining relations is verified directly. The details are omitted. These operations give a convenient method for describing the behavior of Θ_n in the next section.

The face, and degeneracy operations for $F[S^1]$ are prescribed as follows.

$$d_i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ 1 & \text{if } i = j, \text{ and} \\ x_{j-1} & \text{if } j > i. \end{cases}$$

8. The Lie algebra associated to the descending central series for P_{n+1}

The next theorem gives the structure of the Lie algebra arising from the descending central series for P_k which was analyzed in work of T. Kohno [19, 20], Falk, and Randell [15], and Xicoténcatl [32]. Let $B_{i,j}$ denote the projections of the $A_{i,j}$ to $E_0^*(P_k)$.

Theorem 8.1. *The Lie algebra obtained from the descending central series for P_k is given by $\mathcal{L}_k$ the free Lie algebra generated by elements $B_{i,j}$ with $1 \leq i < j \leq k$, modulo the infinitesimal braid relations:*

   (i): $[B_{i,j}, B_{s,t}] = 0$ *if* $\{i,j\} \cap \{s,t\} = \phi$,
   (ii): $[B_{i,j}, B_{i,t} + B_{t,j}] = 0$ *if* $1 \leq i < t < j \leq k$, *and*
   (iii): $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ *if* $1 \leq i < t < j \leq k$.

Furthermore there is a split short exact sequence of Lie algebras

$$0 \to E_0^*(F_n) \xrightarrow{E_0^*(i)} E_0^*(P_{n+1}) \xrightarrow{E_0^*(d_n)} E_0^*(P_n) \to 0$$

where $E_0^*(F_n)$ is the free Lie algebra generated by $B_{i,n+1}$ for $1 \leq i < n+1$. In addition, $E_0^*(P_{n+1})$ is additively isomorphic to $E_0^*(P_n) \oplus E_0^*(F_n)$.

A related direct computation which is used below follows.

Proposition 8.2.   (1) *If* $1 \leq j < t < k \leq n+1$, *then*

$$[B_{j,k} + B_{t,k}, B_{j,t}] = 0,$$

   and

$$[B_{j,t}, B_{t,k}] = [B_{t,k}, B_{j,k}].$$

(2) *If $1 \le r < m < n+1$, then*

$$[\sum_{1 \le i \le n} B_{i,n+1}, \sum_{1 \le j \le r} B_{j,m}] = 0.$$

Proof. Assume that $1 \le j < t < k \le n+1$, and consider the infinitesimal braid relations as follows:

- $[B_{j,t} + B_{j,k}, B_{t,k}] = 0$
- $[B_{j,t} + B_{t,k}, B_{j,k}] = 0$

Thus $[B_{j,k} + B_{t,k}, B_{j,t}] = [B_{t,k}, B_{j,k}] + [B_{j,k}, B_{t,k}] = 0$.

In addition, $[B_{j,t} + B_{j,k}, B_{t,k}] = 0$, and thus $[B_{j,t}, B_{t,k}] = [B_{t,k}, B_{j,k}]$.

Consider $[\sum_{1 \le i \le n} B_{i,n+1}, B_{j,m}]$ for $m < n+1$. Since $[B_{i,n+1}, B_{j,m}] = 0$ if $\{i, n+1\} \cap \{j, m\} = \phi$, it follows that $[\sum_{1 \le i \le n} B_{i,n+1}, B_{j,m}] = [B_{m,n+1} + B_{j,n+1}, B_{j,m}] = 0$ by the infinitesimal braid relations.

The proposition follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

9. ON $\Theta_n : F_n \to P_{n+1}$

The proof that the map $\Theta_n : F_n \to P_{n+1}$ is a monomorphism depends on the structure of certain Lie algebras given in this section. These Lie algebras arise from passage to the associated graded for the descending central series filtration of a discrete group arising in the commutative diagram of groups

$$
\begin{array}{ccc}
F_n & \xrightarrow{\Theta_n} & P_{n+1} \\
\downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} \\
F_{n-1} & \xrightarrow{\Theta_{n-1}} & P_n.
\end{array}
$$

This diagram together with induction on n is used to prove the following theorem.

Theorem 9.1. *The maps $\Theta_n : F[x_1, x_2, \cdots, x_n] \to P_{n+1}$ on the level of associated graded Lie algebras*

$$E_0^*(\Theta_n) : E_0^*(F[x_1, x_2, \cdots, x_n]) \to E_0^*(P_{n+1})$$

are monomorphisms. Thus, by Theorem 6.2, the maps Θ_n are monomorphisms.

The hypothesis that Θ_1 is an isomorphism gives the initial step in an induction with the assumption that $E_0^*(\Theta_{n-1})$ is an embedding. To carry out the inductive step, notice that there is a commutative diagram of morphism of Lie algebras

$$
\begin{array}{ccc}
E_0^*(F_n) & \xrightarrow{E_0^*(\Theta_n)} & E_0^*(P_{n+1}) \\
\downarrow{\scriptstyle E_0^*(d_n)} & & \downarrow{\scriptstyle E_0^*(d_n)} \\
E_0^*(F_{n-1}) & \xrightarrow{E_0^*(\Theta_{n-1})} & E_0^*(P_n).
\end{array}
$$

Most of this section gives explicit results concerning these Lie algebras which are used to prove Theorem 9.1 in the next section. Thus recall P. Hall's classical result that

$$E_0^*(F_n) = E_0^*(F[x_1, x_2, \cdots, x_n])$$

is isomorphic to the free Lie algebra over the integers $\mathbb{Z}$

$$L[x_1, x_2, \cdots, x_n]$$

where the x_i's in the free Lie algebra are the projections of the analogous elements in the group F_n. Furthermore, the kernel of the projection map

$$\pi : L[x_1, x_2, \cdots, x_n] \to L[x_2, \cdots, x_n]$$

defined by

$$\pi(x_i) = \begin{cases} 0 & \text{if } i = 1, \\ x_i & \text{if } i > 1. \end{cases}$$

is given by $L[S_n]$ where S_n is the set

$$\{x_1, \text{ad}(x_{i_1})\text{ad}(x_{i_2}) \cdots \text{ad}(x_{i_p})(x_1)\}$$

for all $p \geq 1$ with $i_j > 1$. Thus the Lie algebra kernel of

$$E_0^*(d_n) : E_0^*(F_n) \rightarrow E_0^*(F_{n-1})$$

is given by $L[S_n]$.

It is not the case that the exact sequence of groups $1 \rightarrow \ker(d_n) \rightarrow F_n \rightarrow F_{n-1} \rightarrow 1$ induces a short exact sequence of Lie algebras after passage to the sub-quotients of the descending central series. The details of this last assertion are not included.

It is convenient to give define additional elements specified in the following formulae.

(1) $\Lambda_n = B_{1,n+1} + B_{2,n+1} + \cdots + B_{n,n+1}$, and
(2) $\gamma_q(n) = -\Sigma_{n-q+2 \leq i \leq n} B_{i,n+1}$ if $2 \leq q \leq n$.

Some values of $\Theta(x_i)$ are required next, and are recorded in the next theorem which is a direct computation.

Theorem 9.2. *The map* $\Theta_n : F[x_1, x_2, \cdots, x_n] \rightarrow P_{n+1}$ *satisfies the following formula:*

$$E_0^*(\Theta_n)(x_q) = \Sigma_{1 \leq i \leq n-q+1 < j \leq n+1} B_{i,j}$$

on the level of associated graded Lie algebras $E_0^*(P_{n+1})$ *for* $1 \leq q \leq n$. *In addition, the following formulas hold.*

(1) $E_0^*(\Theta_n)(x_1) = \Lambda_n$.
(2) *If* $j < n + 1$, $[\Lambda_n, B_{i,j}] = 0$.
(3) $E_0^*(\Theta_n)(x_q) = \Lambda_n + \gamma_q(n) + \Sigma_{1 \leq i \leq n-q+1 < j \leq n} B_{i,j}$.
(4) *If* $2 \leq p \leq q$, *then* $[\gamma_p(n), \Theta(x_q(n))] = 0$.

Proof. The formula

$$E_0^*(\Theta_n)(x_q) = \Sigma_{1 \leq i \leq n-q+1, n-q+2 \leq j \leq n} B_{i,j}$$

is that given by the degeneracies as described in section 6, the simplicial structure for AP_*. The next 4 formulae are proven next with formula 1 given by the definition of Λ_n.

With the condition $s < t < n + 1$, notice that $[B_{i,n+1}, B_{s,t}] = 0$ for $i \neq s, t$. Thus

$$[\Lambda_n, B_{s,t}] = [\Sigma_{1 \leq i \leq n} B_{i,n+1}, B_{s,t}],$$

and

$$[\Lambda_n, B_{s,t}] = [B_{s,n+1} + B_{t,n+1}, B_{s,t}]$$

which is 0 by the infinitesimal braid relations in Proposition 8.2. Formula 2 follows.

Since

$$E_0^*(\Theta_n)(x_q) = \Sigma_{1 \leq i \leq n-q+1, n-q+2 \leq j \leq n} B_{i,j},$$

and

$$\Lambda_n + \gamma_q(n) = \Sigma_{1 \leq i \leq n-q+1} B_{i,n+1},$$

formula 3 that $E_0^*(\Theta_n)(x_q) = \Lambda_n + \gamma_q(n) + \Sigma_{1 \leq i \leq n-q+1 < j \leq n} B_{i,j}$ follows directly.

To work out part 4, consider

$$[\gamma_p(n), \Theta(x_q)] = -[\Sigma_{n-p+1 \leq i \leq n} B_{i,n+1}, \Sigma_{1 \leq i \leq n-q+1 < j \leq n+1} B_{i,j}].$$

Then expand each term for $n - p + 1 \leq i \leq n$ given by

$$[B_{i,n+1}, \Sigma_{1 \leq i \leq n-q+1 < \leq j \leq n+1} B_{i,j}].$$

There are two cases to check.

(1) If $i < n - q + 1$, then $[B_{i,n+1}, \Sigma_{1 \leq i \leq n-q+1 < \leq j \leq n+1} B_{i,j}] = 0$ by the infinitesimal braid relations.

(2) If $i = n - q + 1$, then $[B_{n-q+1,n+1}, \Sigma_{1 \leq i \leq n-q+1 < \leq j \leq n+1} B_{i,j}] = 0$ by the infinitesimal braid relations.

The theorem follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Some additional properties concerning the degeneracies are given next follow at once.

Theorem 9.3. *The map $\Theta_n : F[x_1, x_2, \cdots, x_n] \rightarrow P_{n+1}$ satisfies the following formula:*

(1) *For any fixed $0 \leq j \leq n$, $s_j \Lambda_n = \Lambda_{n+1}$.*

(2)

$$s_j(\gamma_q(n)) = \begin{cases} \gamma_q(n+1) & \text{if } j < n+1-q, \\ \gamma_{q+1}(n+1) & \text{if } j \geq n+1-q. \end{cases}$$

(3) $[\gamma_3(3), \Theta(x_2)] = [\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)]$.

(4) *For any fixed $2 \leq i < j \leq n$, there are sequences of degeneracies $s(I, J)$ such that*

- $s(I, J)(x_2) = x_i$,
- $s(I, J)(x_3) = x_j$,
- $s(I, J)([\gamma_3(3), E_0^*(\Theta_n)(x_2)] = [\gamma_j(n), E_0^*(\Theta_n)(x_i)]$.

(5) *For any fixed $2 \leq i < j \leq n$,*

$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)].$$

(6) *If $2 \leq p \leq q$, then*

$$[\gamma_p(n), \Theta_n(x_q(n))] = 0.$$

(7) *If $2 \leq q < p$, then*

$$[\gamma_p(n), \Theta_n(x_q(n)) - 2\gamma_q(n)] = [\gamma_p(n), \Lambda_n] + [\Lambda_n, \gamma_q(n)].$$

Proof. The degeneracies are described in section 7 which gives the simplicial structure for AP_*. Recall that $\Lambda_n = \Sigma_{1 \leq i \leq n} B_{i,n+1}$. That $s_j \Lambda_n = \Lambda_{n+1}$, and formula 1 follows at once. Recall that $\gamma_q(n) = B_{n+2-q,n+1} + B_{n+3-q,n+1} + \cdots + B_{n,n+1}$. Thus if $j \leq n+1-q$, then $s_j \gamma_q(n) = B_{n+3-q,n+2} + B_{n+4-q,n+2} + \cdots + B_{n+1,n+2}$. Furthermore, if $n+2-q \leq j$, then $s_j \gamma_q(n) = B_{n+2-q,n+2} + B_{n+3-q,n+2} + \cdots + B_{n+1,n+2}$, and formula 2 follows.

To check formula 3, notice that the following hold.

(1) By definition, $\gamma_3(3) = -(B_{2,4} + B_{3,4})$.

(2) By definition, $E_0^*(\Theta_3)(x_2) = B_{1,3} + B_{2,3} + B_{1,4} + B_{2,4}$.

(3) Thus $[\gamma_3(3), E_0^*(\Theta_3)(x_2)] = -[B_{2,4} + B_{3,4}, B_{1,3} + B_{2,3} + B_{1,4} + B_{2,4}]$ which equals

(4) $-[B_{2,4}, B_{1,3} + B_{2,3} + B_{1,4} + B_{2,4}]$ by the infinitesimal braid relations.

(5) Thus $[\gamma_3(3), E_0^*(\Theta_3)(x_2)] = -[B_{2,4}, B_{2,3} + B_{1,4}]$ by the infinitesimal braid relations.

(6) Furthermore, $[B_{2,4}, B_{2,3} + B_{3,4}] = 0$ by the infinitesimal braid relations, and thus

(7) $[\gamma_3(3), \Theta(x_2)] = [B_{2,4}, B_{3,4}] - [B_{2,4}, B_{1,4}] = [B_{2,4}, B_{3,4}] - [B_{2,4}, \Lambda_3 - B_{3,4}]$.

(8) Substituting $\gamma_2(3) = -B_{3,4}$, $\gamma_3(3) = -(B_{2,4} + B_{3,4}$, and $\gamma_2(3) - \gamma_3(3) = B_{2,4}$ gives $[\gamma_3(3), \Theta(x_2)] = [\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)]$. Thus formula 3 follows.

Formula 4 clearly follows from the degeneracies.

To check formula 5, notice that

$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = s(I, J)([\gamma_3(3), E_0^*(\Theta_n)(x_2)],$$

$$s(I, J)([\gamma_3(3), E_0^*(\Theta_3)(x_2)] = S(I, J)([\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)]),$$

and

$$S(I, J)([\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)])$$

equal to

$$[\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)].$$

Formula 6 is formula 4 in Theorem 9.2.

By formula 5, For any fixed $2 \leq i < j \leq n$,

$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)].$$

Thus

$$[\gamma_j(n), E_0^*(\Theta_n)(x_i) - 2\gamma_i(n)] = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)],$$

and formula 7 follows.

$\square$

10. ON THE PROOF OF THEOREM 9.1

Recall the morphism of Lie algebras

$$
\begin{array}{ccc}
E_0^*(F_n) & \xrightarrow{E_0^*(\Theta_n)} & E_0^*(P_{n+1}) \\
\downarrow {\scriptstyle E_0^*(d_n)} & & \downarrow {\scriptstyle E_0^*(d_n)} \\
E_0^*(F_{n-1}) & \xrightarrow{E_0^*(\Theta_{n-1})} & E_0^*(P_n).
\end{array}
$$

Taking Lie algebra kernels results in a commutative diagram

$$
\begin{array}{ccc}
L[x_1^{E_0^*(F_{n-1})}] & \xrightarrow{E_0^*(\Theta_n)} & L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}] \\
\downarrow & & \downarrow \\
E_0^*(F_n) & \xrightarrow{E_0^*(\Theta_n)} & E_0^*(P_{n+1}) \\
\downarrow {\scriptstyle E_0^*(d_n)} & & \downarrow {\scriptstyle E_0^*(d_n)} \\
E_0^*(F_{n-1}) & \xrightarrow{E_0^*(\Theta_{n-1})} & E_0^*(P_n)
\end{array}
$$

for which

- the Lie algebra kernel of $E_0^*(d_n) : E_0^*(F_n) \to E_0^*(F_{n-1})$ is $L[x_1^{E_0^*(F_{n-1})}]$, and
- the Lie algebra kernel of $E_0^*(d_n) : E_0^*(P_{n+1}) \to E_0^*(P_n)$ is

$$L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}].$$

The inductive hypothesis is that $E_0^*(\Theta_{n-1})$ is a monomorphism. To finish, it suffices to check that the induced map of Lie algebras

$$L[x_1^{E_0^*(F_{n-1})}] \xrightarrow{E_0^*(\Theta_n)} L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}]$$

is a monomorphism. This will be checked using the computations of section 8.

The next step is to consider the morphism of Lie algebras

$$p : L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}] \to L[B_{2,n+1}, \cdots, B_{n,n+1}]$$

defined by the formula

$$p(B_{j,n+1}) = \begin{cases} B_{j,n+1} & \text{if } j > 1, \\ -(B_{2,n+1} + B_{3,n+1} + \cdots + B_{n,n+1}) & \text{if } j = 1. \end{cases}$$

Notice that $p(\Sigma_{1 \leq j \leq n} B_{j,n+1}) = 0$, and so

$$p(\Lambda_n) = 0.$$

A remark is appropriate here. With the exception of the map p, all of the above maps are morphisms of simplicial Lie algebras. The map p fails to preserve one face operation, and is thus not a morphism of simplicial Lie algebras.

Let B equal $E_0^*(F_{n-1})$. The image of

$$E_0^*(\Theta_n) : L[x_1^B] \rightarrow L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}]$$

is in the Lie ideal generated by $\Theta(x_1) = \Lambda_n$, and thus is in the Lie algebra kernel of p. Hence the map Θ restricts to a map of Lie algebras

$$E_0^*(\Theta_n) : L[x_1^B] \rightarrow L[\Lambda_n^C]$$

where $C = L[B_{2,n+1}, B_{3,n+1}, \cdots, B_{n,n+1}]$. In the work which is given below, the notation $x_j = x_j(n)$, and $\gamma_j = \gamma_j(n)$ is used when the integer n is clear from the context.

The structure of the Lie algebras $L[x_1^B]$, and $L[\Lambda_n^C]$ are used below. Recall that the abelianization of the Lie algebra $L[S]$ is equal to

$$H_1(L[S]) = L[S]/([L[S], L[S]]).$$

Some of this structure is well-known, and recorded next.

Theorem 10.1. *The abelianization of $L[x_1^B]$, $H_1 L[x_1^B]$, is given by the free abelian group with basis x_1, and $[\cdots [x_1, x_{j_1}] x_{j_2}] \cdots] x_{j_{q-1}}] x_{j_q}]$ for all sequences $2 \leq j_1, j_2, \cdots, j_q$.*

The abelianization of $L[\Lambda_n^C]$, $H_1 L[\Lambda_n^C]$, is given by the free abelian group with basis Λ_n, and $[\cdots [\Lambda_n, \gamma_{j_1}] \gamma_{j_2}] \cdots] \gamma_{j_{q-1}}] \gamma_{j_q}]$ for $2 \leq j_1, j_2, \cdots, j_q$.

The graded free abelian groups $H_1 L[x_1^B]$, and $H_1 L[\Lambda_n^C]$ are filtered as described next for which the filtration $H_1 L[\Lambda_n^C]$ satisfies a reversed ordering from that of $H_1 L[x_1^B]$. This reversal is forced by the infinitesimal braid relations, and the requirement that the map $E_0^*(\Theta_n)$ preserve filtrations on the level of the first homology group as suggested by Theorems 10.4, and 10.7 below.

Definition 10.2. Grade the first homology group $H_1(L[x_1^B])$ by setting

$$Gr_q(H_1(L[x_1^B]))$$

for $q \geq 1$ equal to the linear span of the classes

$$[\cdots [x_1, x_{j_1}] x_{j_2}] \cdots] x_{j_{q-1}}] x_{j_q}]$$

for all sequences $2 \leq j_1, j_2, \cdots j_{q-1}, j_q$ with the convention that $Gr_1(H_1(L[x_1^B]))$ is the linear span of of the class x_1. Thus

$$H_1(L[x_1^B]) = \oplus_{q \geq 1} Gr_q(H_1(L[x_1^B])).$$

Filter $Gr_q(H_1(L[x_1^B]))$ by $F_p(Gr_q(H_1(L[x_1^B])))$ as follows.

(1) For every $q \geq 1$,
$$F_0 Gr_q(H_1(L[x_1^B]))$$
is the linear span of the classes
$$[\cdots [x_1, x_{j_1}] x_{j_2}] \cdots] x_{j_{q-1}}] x_{j_q}]$$
for all sequences
$$2 \leq j_1 \leq j_2 \leq \cdots \leq j_q.$$

(2) Filtration $p > 0$, $F_p(Gr_q(H_1(L[x_1^B])))$, is the linear span of the classes
$$[\cdots [x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_q}]x_{j_q}]$$
with $D([\cdots [x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_q}]x_{j_q}]) \leq p$ where
$$D([\cdots [x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_q}]x_{j_q}]) = \Sigma_{1 \leq i \leq q-1} d(x_{j_i})$$
with $d(x_{j_i})$ equal to the number of $x_{j_{i+k}}$ such that $j_i > j_{i+k}$ for $k > 0$.

Thus, there are inclusions
$$F_p(Gr_q(H_1(L[x_1^B]))) \subset F_{p+1}(Gr_q(H_1(L[x_1^B])))$$
for all p, and
$$Gr_q(H_1(L[x_1^B])) = \cup_{0 \leq p} F_p(Gr_q(H_1(L[x_1^B]))).$$

There is a similar filtration $H_1 L[\Lambda_n^C]$, but with reversed ordering.

Definition 10.3. Grade the first homology group
$$H_1(L[\Lambda_n^C])$$
by setting
$$Gr_q(H_1(L[\Lambda_n^C])$$
for $q \geq 1$ equal to the linear span of the classes
$$[\cdots [\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]$$
for all sequences $2 \leq j_1, j_2, \cdots j_{q-1}, j_q$ with the convention that $Gr_1(H_1(L[\Lambda_n^C])$ is the linear span of the class x_1. Thus
$$H_1(L[\Lambda_n^C]) = \oplus_{q \geq 1} Gr_q(H_1(L[\Lambda_n^C])).$$
Filter $Gr_q(H_1(L[\Lambda_n^C])$ by $F_p(Gr_q(H_1(L[\Lambda_n^C])))$ as follows.
   (1) For every $q \geq 1$,
$$F_0 Gr_q(H_1(L[\Lambda_n^C])$$
   is the linear span of the classes
$$[\cdots [\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]$$
   for all sequences
$$j_1 \geq j_2 \geq \cdots \geq j_q \geq 2.$$
   (2) Filtration $p > 0$, $F_p(Gr_q(H_1(L[\Lambda_n^C])))$ is the linear span of the classes
$$[\cdots [\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]$$
   such that
$$\Delta([\cdots [\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]) = \Sigma_{1 \leq i \leq q-1} \delta(\gamma_{j_i})$$
   with $\delta(\gamma_{j_i})$ equal to the number of $\gamma_{j_{i+k}}$ such that $j_i < j_{i+k}$ for $k > 0$.

Thus, there are inclusions
$$F_p(Gr_q(H_1(L[\Lambda_n^C]))) \subset F_{p+1}(Gr_q(H_1(L[\Lambda_n^C])))$$
for all p, and
$$Gr_q(H_1(L[\Lambda_n^C])) = \cup_{0 \leq p} F_p(Gr_q(H_1(L[\Lambda_n^C]))).$$

Theorem 10.4. *The map of Lie algebras $E_0^*(\Theta_n) : L[x_1^B] \to L[\Lambda_n^C]$ induces a map*
$$H_1(\Theta_n) : H_1 L[x_1^B] \to H_1 L[\Lambda_n^C]$$
which
   (1) *preserves both gradation, and filtration as given in Definitions 10.2, and 10.3, and*
   (2) *is an isomorphism.*

Corollary 10.5. *The map of Lie algebras $E_0^*(\Theta_n) : L[x_1^B] \to L[\Lambda_n^C]$ is an isomorphism.*

The proof of Theorem 10.4 follows by induction with the first case given next in which $x_i = x_i(n)$, and $\gamma_i = \gamma_i(n)$.

Theorem 10.6. *If $n \geq 1$, then $E_0^*(\Theta_n)(x_1) = \Lambda_n$. If $2 \leq j_1 \leq j_2 \leq \cdots \leq j_q$, then*

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]) = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}].$$

Thus $E_0^*(\Theta_n)$ preserves filtration 0 in Theorem 10.4, and induces an isomorphism

$$H_1(\Theta_n) : F_0(Gr_q(H_1(L[x_1^B]))) \to F_0(Gr_q(H_1(L[\Lambda_1^C])))$$

for all $q \geq 0$.

Proof. By Theorem 9.2, $E_0^*(\Theta_n)(x_1) = \Lambda_n$. Thus the theorem is correct in the case of the empty sequence. The next step is to check that $E_0^*(\Theta_n)$ preserves filtration 0 by inducting on q as stated in Theorem 10.4. Assume by induction that if $2 \leq j_1 \leq j_2 \leq \cdots \leq j_q$, then

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]) = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}].$$

Next, consider $j_q \leq j_{q+1}$, together with the value of

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}])$$

given by

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]E_0^*(\Theta_n)(x_{q+1})].$$

If $2 \leq j_p \leq j_q$, then $[\gamma_{j_p}(n), \Theta(x_{j_q}(n))] = 0$ by Theorem 9.2. Hence by the Jacobi identity, $[[A, B], C] = [[A, C], B] + [A, [B, C]]$, the value of

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]E_0^*(\Theta_n)(x_{q+1})]$$

is equal to

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]E_0^*(\Theta_n)(x_{q+1})]\gamma_{j_1}].$$

Notice that by the inductive hypothesis, the element

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]E_0^*(\Theta_n)(x_{q+1})]$$

is equal to

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_2}]x_{j_3}]\cdots]x_{j_{q-1}}]x_{j_{q+1}}]).$$

That is, the element x_{j_1} does not appear.

Furthermore, notice that the element $E_0^*(\Theta_n)([\cdots[x_1, x_{j_2}]x_{j_3}]\cdots]x_{j_{q-1}}]x_{j_{q+1}}])$ is equal to

$$[\cdots[\Lambda_n, \gamma_{j_{q+1}}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_3}]\gamma_{j_2}]$$

by the inductive hypothesis. Hence

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}]) = [\cdots[\Lambda_n, \gamma_{j_{q+1}}]\gamma_{j_q}]\cdots]\gamma_{j_2}]\gamma_{j_1}].$$

The theorem follows.   □

The proof of the next theorem is analogous, but uses a variation of Theorem 9.3 to measure changes of order, and their effect on the filtration defined above. To keep track of certain signs, the following conventions are used.

Theorem 10.7. *If $2 \leq j_1, j_2, \cdots, j_q$, then the class of*

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}])$$

in $H_1 L[\Lambda_n^C]$ is equal to the class of

$$(\pm 1)[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega$$

where Ω projects to an element of lower filtration degree in $H_1 L[\Lambda_n^C]$. In addition, $E_0^*(\Theta_n)$ both preserves the filtrations as given in Definitions 10.2, and 10.3 , as well induces an isomorphism on the level of first homology groups.

Proof. The proof of Theorem 10.7 is by induction on filtration degree p for fixed gradation $Gr_q(H_1(L[x_1^B])$ starting with filtration degree $p = 0$. Notice that if $n \geq 1$, then $E_0^*(\Theta_n)(x_1) = \Lambda_n$, and if $2 \leq j_1 \leq j_2 \leq \cdots \leq j_q$, then

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]) = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]$$

by 10.6. Thus the theorem is correct for filtration degree 0, and for all q as the map $E_0^*(\Theta_n)$ induces an isomorphism on the level of

$$H_1(\Theta_n) : F_0 Gr_q(H_1(L[x_1^B])) \rightarrow F_0 Gr_q(H_1(L[\Lambda_n^C]))$$

for all $1 \leq q$.

Thus assume inductively that

$$E_0^*(\Theta_n)[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$$

in $H_1 L[\Lambda_n^C]$ is equal to $(\pm 1)[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega$ where Ω is of lower filtration as stated in Theorem 10.7.

Consider the following formulae.

(1) $E_0^*(\Theta_n)[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}] = [[A, B], C]$ for which

$$A = E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}])$$

with

$$B = E_0^*(\Theta_n)(x_{j_q}),$$

and

$$C = E_0^*(\Theta_n)(x_{j_{q+1}}).$$

(2) The inductive hypothesis gives that

$$[A, B] = (\pm 1)[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega$$

for which Ω projects to lower filtration degree in $Gr_q(H_1(L[x_1^B]))$. Hence

$$[[A, B], C] = [((\pm 1)[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega)], C]$$

in $Gr_q(H_1(L[x_1^C]))$.

(3) Since Ω has lower filtration degree than $[A, B]$, the filtration degree of $[\Omega, C]$ has filtration degree strictly less than that of $[[A, B], C]$ in $Gr_q(H_1(L[\Lambda_n^C])$ by inspection of the definition.

(4) Next consider

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}], C] = [[E, F]C] = [[E, C]F] + [E[F, C]]$$

for which
- $E = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]$, and
- $F = \gamma_{j_{q-1}}$.

(5) Notice that

$$[E, C] = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]E_0^*(\Theta_n)(x_{j_{q+1}})].$$

(6) Thus $[E, C]$ is the sum

$$[[X, E_0^*(\Theta_n)(x_{j_{q+1}})]\gamma_{j_2}] + [X, [\gamma_{j_2}, E_0^*(\Theta_n)(x_{j_{q+1}})]]]$$

where

$$X = \cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_3}].$$

(7) If
$$j_2 \leq j_{q+1},$$
then $[\gamma_{j_2}, E_0^*(\Theta_n)(x_{j_{q+1}})] = 0$ by part 6 of Theorem 9.3. Hence
$$[E, C] = [[X, E_0^*(\Theta_n)(x_{j_{q+1}})]\gamma_{j_2}],$$
the inductive hypothesis applies, and the Theorem follows.

(8) If
$$j_2 > j_{q+1},$$
then
$$[\gamma_{j_2}, E_0^*(\Theta_n)(x_{j_{q+1}})] = V + Y$$
where $V = [\gamma_j(2), \Lambda_n] + [\Lambda_n, \gamma_{j_{q+1}}]$, and $Y = 2[\gamma_{j_2}, \gamma_{j_{q+1}}]$ by part 5 of Theorem 9.3. Thus $[[X, E_0^*(\Theta_n)(x_{j_{q+1}})], \gamma_{j_2}] + [X, [\gamma_{j_2}, E_0^*(\Theta_n)(x_{j_{q+1}})]]$ is equal to the coset of
$$[[X, E_0^*(\Theta_n)(x_{j_{q+1}})]\gamma_{j_2}] + [X, 2[\gamma_{j_2}, \gamma_{j_{q+1}}]]]$$
in $H_1(L[x_1^C])$, and thus
$$[[X, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}]$$
in $H_1(L[x_1^C])$ modulo terms of lower filtration.

(9) Furthermore,
$$[[X, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}] = -[[X, \gamma_{j_{q+1}}]\gamma_{j_2}]$$
in case $X = \Lambda_n$, and
$$[[\gamma_s, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}] = 0.$$

(10) Thus if $j_2 > j_{q+1}$, then $[[X, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}]$ is one of the following.
   (a) $-[[X, \gamma_{j_{q+1}}]\gamma_{j_2}]$, or
   (b) $[[X, \gamma_{j_{q+1}}]\gamma_{j_2}]$

(11) In addition,
$$[\Lambda_n, E_0^*(\Theta_n)(x_{j_{q+1}})] = [\Lambda_n, \gamma_{j_{q+1}}]$$
as follows from 9.3, and 9.2.

Formula 5 in Theorem 9.3 is used at this point, and is stated next for the convenience of the reader. For any fixed $2 \leq i < j \leq n$, $[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = S$ where $S = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)]$. This formula will be used in the expansion of $E_0^*(\Theta_n)[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}]$. Thus,
$$[B, C] = [E_0^*(\Theta_n)(x_{j_q}), E_0^*(\Theta_n)(x_{j_{q+1}})] = [\gamma_{j_q}, \Lambda_n] + [\Lambda_n, \gamma_{j_{q+1}}] + 2[\gamma_{j_q}, \gamma_{j_{q+1}}].$$

Formula 5 in Theorem 9.3 is used at this point, and is stated next for the convenience of the reader. For fixed i, and j with $2 \leq i < j \leq n$,
$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = U + V$$
where $U = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)]$, and $V = 2[\gamma_j(n), \gamma_i(n)]$.

The previous formula will be used in the expansion of the element specified by an expansion of $E_0^*(\Theta_n)([X_q, x_{j_{q+1}}])$ for $X_q = [\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$. Thus, $[B, C] = [E_0^*(\Theta_n)(x_{j_q}), E_0^*(\Theta_n)(x_{j_{q+1}})] = U" + V"$ where $U" = [\gamma_{j_q}, \Lambda_n] + [\Lambda_n, \gamma_{j_{q+1}}]$, and $V" = 2[\gamma_{j_q}, \gamma_{j_{q+1}}]$.

Next, notice that
$$[A, [\gamma_q, \Lambda_n]],$$
and
$$[A, [\Lambda_n, \gamma_{q+1}]]$$
both project to 0 in $Gr_q(H_1(L[\Lambda_n^C]))$. Hence the class of $[A, [B, C]]$ equal to the class of $[A, 2[\gamma_{j_q}, \gamma_{j_{q+1}}]]$.

□

Theorem 10.8. *The map*

$$E_0^*(\Theta_n) : L[x_1^B] \;\to\; L[\Lambda_n^C]$$

induces an isomorphism of Lie algebras, and thus a monomorphism. Hence Θ_n is a monomorphism.

Proof. Notice that the map $E_0^*(\Theta_n)$ sends a generator $[\cdots [x_1, x_{j_1}] x_{j_2}] \cdots] x_{j_{q-1}}] x_{j_q}]$ to

$$(\pm 1)[\cdots [\Lambda_n, \gamma_{j_q}] \gamma_{j_{q-1}}] \cdots] \gamma_{j_2}] \gamma_{j_1}] + \Omega$$

where Ω projects to an element of lower filtration degree in $H_1 L[\Lambda_n^C]$ by Theorem 10.7. Hence the morphism of Lie algebras $E_0^*(\Theta_n)$ induces an isomorphism on the module of indecomposables, and thus an isomorphism of Lie algebras. The Theorem follows.   □

11. On Vassiliev invariants, the mod-p descending central series, and the Bousfield-Kan spectral sequence

Let $\Gamma^n(G)$, respectively $\Gamma^{n,p}(G)$, denote the n-th stage of the descending central series for a discrete group G, respectively, the mod-p descending central series for G. Thus

(1) $\Gamma^n(G)$ is the subgroup of G generated by commutators $[\cdots [g_1, g_2], g_3], \cdots], g_t]$ for $t \geq n$ with decreasing filtration

$$G = \Gamma^1(G) \supseteq \Gamma^2(G) \supseteq \cdots,$$

and

(2) $\Gamma^{n,p}(G)$ is the subgroup of G generated by commutators $[\cdots [g_1, g_2], g_3], \cdots], g_s]^{p^j}$ for $s \cdot p^j \geq n$ with decreasing filtration

$$G = \Gamma^{1,p}(G) \supseteq \Gamma^{2,p}(G) \supseteq \cdots.$$

Let $E_0^n(G) = \Gamma^n(G)/\Gamma^{n+1}(G)$, and $E_0^{n,p}(G) = \Gamma^{n,p}(G)/\Gamma^{n+1,p}(G)$. The commutator map of sets

$$[-, -] : G \times G \to G$$

induces a natural pairing endowing

$$E_0^*(G) = \oplus_{n \geq 1} E_0^n(G)$$

with the structure of a Lie algebra while the analogous pairing for

$$E_0^{*,p}(G) = \oplus_{n \geq 1} E_0^{n,p}(G)$$

with the p-th power map $\xi : G \to G$ which induces a function

$$\xi : E_0^{n,p}(G) \to E_0^{pn,p}(G)$$

gives the structure of a restricted Lie algebra [18].

Results of Kohno-Falk-Randell used above give the structure of $E_0^n(P_{n+1})$ as well as a relationship between the Lie algebra obtained from the descending central series of a group with certain natural choices of group extensions. The variation given in this section replaces the descending central series by the mod-p descending central series thus giving the analogous structure for $E_0^{*,p}(P_{n+1})$.

Recall that the unstable Adams spectral sequence is that obtained by filtering a simplicial group by the mod-p descending central series [11]. This variation is recorded here as it corresponds to the relationship between Vassiliev invariants, the Bousfield-Kan spectral sequence, or unstable Adams spectral sequence.

First, consider the integral version gotten by filtering by the descending central series which is labelled the Bousfield-Kan spectral sequence below. On the level of E_0^*, the morphism of simplicial groups $\Theta : F[S^1] \to \text{AP}_*$ induces a map

$$E_0^*(\Theta_n) : E_0^*(F_n) \to E_0^*(P_{n+1}),$$

the subject of sections 10, and 9 on embeddings of Lie algebras here. In addition, the associated graded Lie algebra for the mod-p descending central series $E_0^{*,p}(F_n)$ gives the E_0-term of a spectral sequence abutting to the homotopy groups of $F[S^1]$ modulo torsion prime to p.

On the other hand, the Lie algebra $E_0^*(P_{n+1})$ was studied in [19, 20, 21] where the universal enveloping algebra was shown to give Vassiliev invariants of pure braids. The Vassiliev invariants distinguish pure braids as proven in [19]. Thus they distinguish elements in the E^0-term of the Bousfield-Kan spectral sequence for $F[S^1]$. The analogue for the mod-p descending central series is developed next.

Question: Is there a further relationship between the Vassiliev invariants of pure braids, and the homotopy groups of the 2-sphere ? Is there an informative interplay between these invariants, and homotopy theory ?

The point of the next remarks is to record the structure of the Lie algebra obtained from the mod-p descending central series for the pure braid groups. Algebraic preliminaries are given next arising from work of [19, 15, 32], and the modification below for the mod-p descending central series.

Theorem 11.1. *Let*

$$1 \to A \xrightarrow{\ \ j\ \ } B \xrightarrow{\ \ p\ \ } C \to 1$$

be a short exact sequence of groups such that

- *there is a section σ for $p : B \to C$ giving $p \circ \sigma = 1_C$, and*
- *the natural action of C on $H_1(A)$ is trivial: Given b in B and a in A, then*

$$bab^{-1} = ax$$

  for some x in the commutator subgroup $[A, A] = \Gamma^2(A)$.

Then there is

(1) *a split short exact sequences of Lie algebras*

$$0 \to E_0^*(A) \to E_0^*(B) \to E_0^*(C) \to 0,$$

   and

(2) *a split short exact sequences of restricted Lie algebras*

$$0 \to E_0^{*,p}(A) \to E_0^{*,p}(B) \to E_0^{*,p}(C) \to 0.$$

The structure of the Lie algebra for the mod-p descending central series of the pure braid group, as well as certain other groups follows from the proof of the Proposition 11.2. A proof is analogous to the ones in [19, 15, 32]; modifications in the case of the mod-p descending central series are direct, and are listed below for convenience.

Observe that $p((b \cdot \sigma(p(b^{-1}))) = 1$, and so there exists an unique element a in A with $j(a) = b \cdot \sigma p(b^{-1})$. Thus, there is a well-defined function (not necessarily a homomorphism)

$$\tau : B \to A$$

defined by the formula

$$\tau(b) = j^{-1}(b \cdot \sigma(p(b^{-1}))).$$

A useful variation of 11.1 for the mod-p descending central series is recorded next.

Proposition 11.2. *Let*

$$1 \to A \xrightarrow{\ \ j\ \ } B \xrightarrow{\ \ p\ \ } C \to 1$$

be a short exact sequence of groups such that

- *there is a section σ for $p : B \to C$ giving $p \circ \sigma = 1_C$, and*
- *the natural action of C on $H_1(A)$ is trivial: Given b in B and a in A, then*

$$bab^{-1} = ax$$

for some x in the commutator subgroup $[A, A] = \Gamma^2(A)$. (Note that x is always in $[B, B]$, but x is not necessarily an element in $[A, A]$.)

Then the following hold.

(1) *For fixed b in B, a in A, there is an element y in $[A, A]$ such that $b^{-1}ab = ay$.*

(2) *The group $[B, A]$ is a subgroup of $[A, A]$.*

(3) *The group $[\Gamma^m(B), \Gamma^n(A)]$ is a subgroup of $\Gamma^{n+m}(A)$, and the group $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m,p}(A)$.*

(4) *If $n \geq 1$, τ is a filtration preserving function (not necessarily a group homomorphism). That is, $\tau(\Gamma^n(B))$ is contained in $\Gamma^n(A)$, and $\tau(\Gamma^{n,p}(B))$ is contained in $\Gamma^{n,p}(A)$.*

(5) *If $p(b) = 1$, then $j\tau(b) = b$. Furthermore, there are split short exact sequences of groups*

$$1 \to \Gamma^n(A) \to \Gamma^n(B) \to \Gamma^n(C) \to 1,$$

and

$$1 \to \Gamma^{n,p}(A) \to \Gamma^{n,p}(B) \to \Gamma^{n,p}(C) \to 1.$$

(6) *If $n \geq 1$, there are well-defined induced functions $\overline{\tau} : E_0^n(B) \to E_0^n(A)$, and $\overline{\tau} : E_0^{n,p}(B) \to E_0^{n,p}(A)$ defined on an equivalence class of b, $[b]$, by the formula $\overline{\tau}([b]) = [\tau(b)]$.*

(7) *If $n \geq 1$, and $[b]$ is in the kernel of the induced homomorphism $E_0^n(p) : E_0^n(B) \to E_0^n(C)$, respectively $[b]$ is in the kernel of the induced homomorphism $E_0^{n,p}(p) : E_0^{n,p}(B) \to E_0^{n,p}(C)$, then $E_0^n(j)(\overline{\tau}[b]) = [b]$, respectively $E_0^{n,p}(j)(\overline{\tau}[b]) = [b]$.*

Furthermore, there are split short exact sequences of abelian groups

$$0 \to E_0^n(A) \to E_0^n(B) \to E_0^n(C) \to 0,$$

and

$$0 \to E_0^{n,p}(A) \to E_0^{n,p}(B) \to E_0^{n,p}(C) \to 0.$$

(8) *The morphisms $0 \to E_0^{*,p}(A) \to E_0^{*,p}(B) \to E_0^{*,p}(C) \to 0$ are of restricted Lie algebras.*

Remark 11.3. *The hypotheses above that $bab^{-1} = ax$ for some x in the commutator subgroup $[A, A]$ is not necessarily satisfied without the hypotheses of trivial local coefficients. Notice that if A is a normal subgroup of B, then $a^{-1}b^{-1}ab$ is always in A, but may not necessarily be in $[A, A]$. An example is given by the group extension $1 \to Z/3Z \to \Sigma_3 \to Z/2Z \to 1$ for which the commutator subgroup $A = Z/3Z$ is trivial, but the group $[B, A]$ is non-trivial for $B = \Sigma_3$. The hypothesis of trivial local coefficients is important in this step.*

Theorem 11.1 is a restatement of Proposition 11.2. The proof of 11.2 is based on the Hall-Witt identities together with another lemma of P. Hall both recorded in the next statement proven in [24], page 290, and [12], page 2. An additional useful statement is also given by 11.4. Recall that $[a, b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$ denotes the commutator of elements a, and b in a group G, and $c^a = c^{-1} \cdot a \cdot c$.

Theorem 11.4. *For any elements of a group G,*

(1) $[a, b] \cdot [b, a] = 1$,

(2) $[a, b \cdot c] = [a, c] \cdot [a, b] \cdot [[a, b], c]$,

(3) $[a \cdot b, c] = [a, c] \cdot [[a, c], b] \cdot [b, c]$,

(4) $[[a, b], c^a] \cdot [[c, a], b^c] \cdot [[b, c], a^b] = 1$,

(5) $[[a, b], c] \cdot [[b, c], a] \cdot [[c, a], b] = [b, a] \cdot [c, a] \cdot [c, b]^a \cdot [a, b] \cdot [a, c]^b \cdot [b, c]^a \cdot [a, c] \cdot [c, a]^b$, *and*

(6) $[a, b^n] = [a, b] \cdot [a, b]^b \cdots [a, b]^{b^{n-1}}$.

If a is an element of $\Gamma^{m,p}(G)$, and b is an element of $\Gamma^{n,p}(G)$, then

$$[a, b^p] = [a, b]^p \cdot z$$

for z in $\Gamma^{(m+pn),p}(G)$.

If A, B, and C are normal subgroups of a group G, then $[[A, B], C]$ is contained in the subgroup generated by $[[B, C], A]$ and $[[C, A], B]$.

The proof of Proposition 11.2 is given next.

Proof. The first statement of the proposition is one of the stated assumptions concerning the extension $1 \to A \to B \to C \to 1$. That is $b^{-1}ab = ax$ for b in B, a in A, and some x in $[A, A]$ by the assumption that the local coefficient system on $H_1(A)$ is trivial.

By part (1), $b^{-1}ab = ax$ for b in B, a in A, and some x in $[A, A]$. Thus $a^{-1}b^{-1}ab = x$, so $[B, A]$ is a subgroup of $[A, A]$. Statement (2) follows.

The proof that $[\Gamma^m(B), \Gamma^n(A)]$ is a subgroup of $\Gamma^{n+m}(A)$ is given in [15, 32]. Modifications in the case of $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ are listed next. Since $\Gamma^1(G) = \Gamma^{1,p}(G)$ for any group G, $[\Gamma^{1,p}(B), \Gamma^{1,p}(A)] = [\Gamma^1(B), \Gamma^1(A)]$ is a subgroup of $\Gamma^2(A)$, and thus $\Gamma^{2,p}(A)$. Consider the case of $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$. Inductively assume that for $m < M$, and $n < N$ that $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m,p}(A)$. That $[\Gamma^{m,p}(B), \Gamma^{n+q,p}(A)]$ is also a subgroup of $\Gamma^{n+m+q,p}(A)$ follows by induction on q via Theorem 11.4, or Appendix A of [12]: If a is an element of $\Gamma^{m,p}(G)$, and b is an element of $\Gamma^{n,p}(G)$, then $[a, b^p] = [a, b]^p \cdot z$ for z in $\Gamma^{(m+pn),p}(G)$ by Theorem 11.4. Assume that b is an element of $\Gamma^{m,p}(B)$, and that a^p is an element of $\Gamma^{n+q-1,p}(A)$, then $[b, a^p]$ is an element of $\Gamma^{m+n+q,p}(A)$.

The other cases are those in [19, 15, 32]: If $G = \Gamma^{m,p}(B)$, $K = A$, $H = \Gamma^{n+q-1,p}(A)$, then $[\Gamma^{m,p}(B), \Gamma^{n+q,p}(A)]$ is a subgroup of $\Gamma^{n+m+q,p}(A)$. A similar argument implies that if $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m,p}(A)$ for $m < M$, and $n < N$, then $[\Gamma^{m+s,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m+s,p}(A)$.

Recall that there is a well-defined function $\tau : B \to A$ given by the formula $\tau(b) = j^{-1}(b \cdot \sigma(p(b^{-1})))$. To prove the fourth statement, notice that the following formula holds which measures the failure of the map τ from being multiplicative:

$$\tau(xy) = (\tau(x))(\tau(y))j^{-1}([(\tau(y))^{-1}, \sigma p(x)]).$$

In addition, $[(\tau(y))^{-1}, \sigma p(x)])$ is an element of $[A, B]$ which is a subgroup of $[A, A]$. Thus for all elements x, y in B,

$$\tau(xy) = (\tau(x))(\tau(y))V,$$

as well as

$$\tau(x^p) = (\tau(x))^p W$$

where V, and W are elements of $[A, A]$ by statement (3).

Let $\lambda = \sigma \circ p$, and observe that

$$\tau(xyx^{-1}y^{-1}) = j^{-1}((xyx^{-1}y^{-1})(\lambda(y^{-1})(\lambda x^{-1})(\lambda x)(\lambda y)).$$

Furthermore, if v is an element of $\Gamma^m(B)$, and $\tau(z)$ lies in $\Gamma^n(A)$, then the commutator $[\tau(z), v]$ lies in $\Gamma^{m+n}(A)$ by part (3). Similarly, if v is an element of $\Gamma^{m,p}(B)$, and $\tau(z)$ lies in $\Gamma^{n,p}(A)$, then the commutator $[\tau(z), v]$ lies in $\Gamma^{m+n,p}(A)$ by part (3). Thus the statement that $\tau(\Gamma^n(B))$ is contained in $\Gamma^n(A)$, and $\tau(\Gamma^{n,p}(B))$ is contained in $\Gamma^{n,p}(A)$ follows by induction on n together with part (3) of the proposition. Statement (4) follows.

Statement (5) concerns $\tau : B \to A$ given by the formula $\tau(b) = j^{-1}(b \cdot \sigma(p(b^{-1})))$. By definition, if $p(b) = 1$, then $p(b^{-1}) = 1$, and $j\tau(b) = b$. Hence if if b is either in $\ker(p) \cap \Gamma^n(B)$, or $\ker(p) \cap \Gamma^{n,p}(B)$, then $\tau(b) = b$. Thus by the preceding remark as well as part (4), τ restricts to functions $\tau|_{\ker(p) \cap \Gamma^n(B)} : \ker(p) \cap \Gamma^n(B) \to \Gamma^n(A)$, and $\tau|_{\ker(p) \cap \Gamma^{n,p}(B)} : \ker(p) \cap \Gamma^{n,p}(B) \to \Gamma^{n,p}(A)$.

It follows that the homomorphisms

$$j : \Gamma^n(A) \to \ker(p) \cap \Gamma^n(B),$$

as well as

$$j : \Gamma^{n,p}(A) \to \ker(p) \cap \Gamma^{n,p}(B)$$

are group isomorphisms. Furthermore, there are exact sequences of groups $1 \to \Gamma^n(A) \to \Gamma^n(B) \to \Gamma^n(C) \to 1$, and $1 \to \Gamma^{n,p}(A) \to \Gamma^{n,p}(B) \to \Gamma^{n,p}(C) \to 1$ which are split by the existence of σ. Part (5) follows.

If $n \geq 1$, there is a well-defined induced function $\bar{\tau} : E_0^n(B) \to E_0^n(A)$ defined on an equivalence class of b by the formula $\bar{\tau}([b]) = \tau(b)$ by parts (c), and (d) together with the formula

$$\tau(xy)(\tau(x))(\tau(y))j^{-1}([(\tau(y))^{-1}, \sigma p(x)]).$$

If the class $[b]$ of an element b in $\Gamma^{n,p}(B)$ is in the kernel of the induced homomorphism $E_0^{n,p}(p) : E_0^{n,p}(B) \to E_0^{n,p}(C)$, then b is in $\Gamma^{n,p}(B)$, and $p(b)$ is in $\Gamma^{n+1,p}(C)$.

Since the natural map $\Gamma^{n+1,p}(p) : \Gamma^{n+1}(B) \to \Gamma^{n+1}(C)$ is a surjection by part (3), there is an element x in $\Gamma^{n+1,p}(B)$ such that $p(x) = p(b)$. Note that b is in $\Gamma^{n,p}(B)$, but x is in $\Gamma^{n+1,p}(B)$.

Hence x^{-1} is in $\Gamma^{n+1,p}(B)$, and bx^{-1} is a representative of the class $[b]$ in $E_0^n(B)$ with $j\tau(bx) = bx$. Hence the natural map $E_0^{n,p}(j) : E_0^{n,p}(A) \to E_0^{n,p}(B)$ is a surjection to the kernel of $\bar{\tau} : E_0^{n,p}(B) \to E_0^{n,p}(C)$, and there is a split short exact sequence $0 \to E_0^{n,p}(A) \to E_0^{n,p}(B) \to E_0^{n,p}(C) \to 0$. Statement (6) as well as the proposition follows from the above. $\qquad\square$

The next theorem follows at once as it was checked in [6], page 251, that the local coefficient system for the extension

$$1 \to F_n \to P_{n+1} \to P_n \to 1$$

is trivial over the integers.

Theorem 11.5. *The restricted Lie algebra obtained from the mod-p descending central series for P_k is given by $\mathcal{L}_{k,p}$ the free restricted Lie algebra over $\mathbb{Z}/p\mathbb{Z}$ generated by elements $B_{i,j}$ with $1 \leq i < j \leq k$, modulo the infinitesimal braid relations:*

(i): $[B_{i,j}, B_{s,t}] = 0$ *if* $\{i,j\} \cap \{s,t\} = \phi$,
(ii): $[B_{i,j}, B_{i,t} + B_{t,j}] = 0$ *if* $1 \leq i < t < j \leq k$, *and*
(iii): $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ *if* $1 \leq i < t < j \leq k$.

Furthermore there is a split short exact sequence of restricted Lie algebras

$$0 \to E_0^{*,p}(F_n) \xrightarrow{E_0^{*,p}(i)} E_0^{*,p}(P_{n+1}) \xrightarrow{E_0^{*,p}(d_n)} E_0^{*,p}(P_n) \to 0$$

where $E_0^{*,p}(F_n)$ is the free restricted Lie algebra generated by $B_{i,n+1}$ for $1 \leq i < n+1$. In addition, $E_0^{*,p}(P_{n+1})$ is additively isomorphic to $E_0^{*,p}(P_n) \oplus E_0^{*,p}(F_n)$.

The following is an "integrality" statement concerning as embeddings of Lie algebras may not induce an embedding after mod-p reduction.

Theorem 11.6. *If $n \geq 1$, the induced maps*

$$E_0^*(\Theta_n) : E_0^*(F_n) \to E_0^*(P_{n+1}),$$

and

$$E_0^{*,p}(\Theta_n) : E_0^{*,p}(F_n) \to E_0^{*,p}(P_{n+1})$$

are monomorphisms.

Proof. Notice that the map $E_0^*(\Theta_n)$ sends a generator $[\cdots [x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$ to

$$(\pm 1)[\cdots [\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega$$

where Ω projects to an element of lower filtration degree in $H_1 L[\Lambda_n^C]$ by Theorem 10.7. By 10.8, the map

$$E_0^*(\Theta_n) : L[x_1^B] \to L[\Lambda_n^C]$$

induces an isomorphism of Lie algebras. Hence, the induced map on the level of universal enveloping algebras is an isomorphism over the integers. Thus, there is an induced isomorphism after reduction modulo p, and an isomorphism on the level of restricted Lie algebras. This suffices, and the Theorem follows.   □

12. On braid groups, and axioms for connected CW-complexes

The purpose of this section is to give axioms which characterize CW-complexes in terms of braid groups when viewed within the context of simplicial groups. First consider the category of groups $\mathcal{G}$, and the category of reduced simplicial groups $\mathcal{SG}$. Let $\mathcal{C}$ denote a small category. The definition of a simplicial subgroup is used next, and is defined below as the authors are unaware of an appropriate reference.

Consider the following axioms: Let $\mathcal{B}$ denote the smallest subcategory of $\mathcal{SG}$ which satisfies the following properties:

(1) The simplicial group AP_* is in $\mathcal{B}$.
(2) If Π, and Γ are in $\mathcal{B}$, then the coproduct $\Pi \vee \Gamma$ is in $\mathcal{B}$.
(3) If Π is in $\mathcal{B}$, and Γ is a simplicial subgroup of Π, then Γ is in $\mathcal{B}$.
(4) If Π is in $\mathcal{B}$, and Γ is a simplicial quotient of Π, then Γ is in $\mathcal{B}$.

The next result is stated in the Introduction as Theorem 1.4.

Theorem 12.1. *Let $X(i)$, $i = 1, 2$ denote path-connected CW-complexes with a continuous function $f : X(1) \to X(2)$. Then there exist elements $\Gamma_{X(i)}$, together with a morphism $\gamma : \Gamma_{X(1)} \to \Gamma_{X(2)}$ in $\mathcal{B}$ such that the loop space $\Omega(X(i))$ is homotopy equivalent to the geometric realization of $\Gamma_{X(i)}$, and the induced map $|\gamma| : |\Gamma_{X(1)}| \to |\Gamma_{X(2)}|$ is homotopic to $\Omega(f)$.*

First, the definition of a simplicial subgroup is required. The authors are unaware of a good reference, so additional features are listed below.

Definition 12.2.   (1) A map $f : G \to H$ in $\mathcal{C}$ is a monomorphism provided whenever there are two maps $\alpha, \beta : \pi \to G$ in $\mathcal{C}$ such that

$$f \circ \alpha = f \circ \beta,$$

then

$$\alpha = \beta.$$

(2) A map $f : G \to H$ in $\mathcal{C}$ is an injection provided f is one-to-one on the underlying sets.

Next recall the following standard fact.

Proposition 12.3. *A map* $f : G \to H$ *in* $\mathcal{G}$ *is an injection if and only if* f *is a monomorphism.*

Proof. Assume that f is a monomorphism. It will be checked that $\ker(f)$ is trivial. Assume that there is some non-identity element x in $\ker(f)$. Define

(1) $\alpha : \mathbb{Z} \to G$ by $\alpha(n) = 1$, and
(2) $\beta : \mathbb{Z} \to G$ by $\beta(n) = x^n$.

Then $f \circ \alpha = f \circ \beta$, but $\alpha(x) \neq \beta(x)$, and so $\alpha \neq \beta$. That is a contradiction, and thus $\ker(f)$ is trivial, and so f is an injection.

Next, assume that f is an injection, and that $f \circ \alpha = f \circ \beta$. Thus $f(\alpha(x)) = f(\beta(x))$. Hence $\alpha(x) = \beta(x)$ and the proposition follows. $\qquad \square$

The next step is to check an analogous statement for reduced simplicial groups. First, two definitions should be given.

Definition 12.4. Let Γ, and Π denote reduced simplicial groups. Then Γ is a simplicial subgroup of Π provided there is a monomorphism $\phi : \Gamma \to \Pi$ in $\mathcal{SG}$.

Definition 12.5. Let Γ, and Π denote reduced simplicial groups. Then the pair (Π, Γ) is a simplicial group pair provided there is a morphism in $\mathcal{SG}$ given by $\phi : \Gamma \to \Pi$ which is a degree-wise injection of groups.

Proposition 12.6. *Given simplicial groups* Γ, *and* Π, *the following are equivalent.*

(1) Γ *is a simplicial subgroup of* Π.
(2) *The pair* (Π, Γ) *is a simplicial group pair.*

Proof. Assume that $f : \Gamma \to \Pi$ is a monomorphism in $\mathcal{SG}$. It will be checked that in each simplicial degree n, $\ker(f)$ is trivial. Assume that n is the minimal degree for which there is a non-trivial element x in $\ker(f)$.

Let

$$\bar{G}\langle x \rangle$$

denote the simplicial closure of x as given in [8]. Thus $\bar{G}\langle x \rangle$ is both a simplicial group, and the natural morphism of simplicial groups $\beta : \bar{G}\langle x \rangle \to \Gamma$ satisfies

$$\beta(x) = x.$$

Thus, there are morphisms in $\mathcal{SG}$ given by

(1) $\alpha : \bar{G}\langle x \rangle \to \Gamma$ by $\alpha(x) = 1$, and
(2) $\beta : \bar{G}\langle x \rangle \to \Gamma$ by $\beta(x) = x$.

Then $f \circ \alpha = f \circ \beta$, but $\alpha(x) \neq \beta(x)$, and so $\alpha \neq \beta$.

This statement contradicts the fact that f is monomorphism. Thus $\ker(f)$ is trivial, and f is an injection in each degree.

Next, assume that f is an injection in each degree, and that $f \circ \alpha = f \circ \beta$. Thus in each degree, it follows that $f(\alpha(x)) = f(\beta(x))$. Hence $\alpha(x) = \beta(x)$ and the proposition follows. $\qquad \square$

13. PROOF OF THEOREM 1.4

The simplicial group AP_* is in $\mathcal{B}$ by axiom 1. By Theorem 1.2, there is a morphism of simplicial groups

$$\Theta : F[S^1] \to \mathrm{AP}_*$$

which is a degree-wise injection. Thus $F[S^1]$ is a simplicial subgroup of AP_* by 12.6. Hence $F[S^1]$ is in $\mathcal{B}$ by axiom 3.

Notice that coproducts, $\mathrm{AP}_* \vee \mathrm{AP}_*$, as well as $F[S^1] \vee F[S^1]$ are in $\mathcal{B}$ by axiom 2. Since $F[S^n]$ is a subgroup of $F[S^1] \vee F[S^1]$ for $n \geq 1$, $F[S^n]$ is in $\mathcal{B}$. By passage to coproducts, $\vee_{n \epsilon T} F[S^n]$ is in $\mathcal{B}$ for any set T.

The next statement concerning push-outs is the simplicial analogue of a classical result of J. H. C. Whitehead [16, 23]. found in [31].

Proposition 13.1. *Let G_0, G_1, and G_2 be simplicial groups in $\mathcal{B}$ together with morphisms $\alpha : G_0 \to G_1$, and $\beta : G_0 \to G_2$ in $\mathcal{B}$. Then*

(1) *the push-out Π of $\alpha : G_0 \to G_1$, and $\beta : G_0 \to G_2$ is in $\mathcal{B}$, and*

(2) *if both α, and β are monomorphisms in $\mathcal{B}$, the classifying space of Π is the push-out of the classifing space construction of $\alpha : G_0 \to G_1$, and $\beta : G_0 \to G_2$.*

Let Y be the cofibre of a map $a : \vee_{n\epsilon T} S^n \to X$, and assume that ΩX is homotopy equivalent to a simplicial group G in $\mathcal{B}$. Then consider the push-out of groups

$$
\begin{array}{ccc}
F[\vee_{n\epsilon T} S^{n-1}] & \xrightarrow{\ f\ } & G \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Theta} \\
\vee_S \mathrm{AP}_* & \longrightarrow & \Gamma
\end{array}
$$

where f is the natural extension of the attaching map a, and g can be chosen to be a monomorphism. If f is a monomorphism, the push-out Γ lies in the category $\mathcal{B}$, then the geometric realization of G is homotopy equivalent to ΩY because Γ is the free product with amalgamation by the subgroup $F[\vee_S S^{n-1}]$, and the coproduct of $\vee_S \mathrm{AP}_*$ is contractible.

It will be checked next that f may be assumed to be a monomorphism. First assume that G' is homotopy equivalent to ΩX and

$$
f' : F[\vee_{n\epsilon T} S^{n-1}] \to G'
$$

represents the looping of the attaching map. Observe that $F[\vee_{n\epsilon T} S^{n-1}]$ embeds in $\vee_S \mathrm{AP}_*$ via the injection

$$
g : F[\vee_{n\epsilon T} S^{n-1}] \to \vee_S \mathrm{AP}_*.
$$

Next, recall that $\mathcal{B}$ contains products as it contains both coproducts, and quotients. Let

$$
G = G' \times (\vee_S \mathrm{AP}_*),
$$

and let f be the injection

$$
f' \times g : F[\vee_{n\epsilon T} S^{n-1}] \to G.
$$

Notice that G is homotopy equivalent to ΩX because AP_* is contractible, and that G is in $\mathcal{B}$. Hence f may be replaced by $f' \times g$, and thus assumed to be a monomorphism.

To verify naturality, the previous arguments apply to continuous maps $f : Y \to Z$ of connected CW complexes with $Y = \vee_{n\epsilon T} S^n$. In addition, if Y be the cofibre of a map $a : \vee_{n\epsilon T} S^n \to X$, an application of the push-out constructed above suffices.

The result follows by 13.1.

14. APPENDIX: A SAMPLE COMPUTATION

The purpose of this section is to list a sample computation for the values of Θ_3.

(1) $\Theta_3([x_1, x_2]x_2]x_3]) = [\Lambda_3, \gamma_3]\gamma_2]\gamma_2]$,

(2) $\Theta_3([x_1, x_2]x_3]x_2]) \cong -[\Lambda_3, \gamma_2]\gamma_3]\gamma_2] + 2[\Lambda_3, \gamma_3]\gamma_2]\gamma_2]$ modulo decomposables, and

(3) $\Theta_3([x_1, x_3]x_2]x_2]) \cong [\Lambda_3, \gamma_2]\gamma_2]\gamma_3] - 2[\Lambda_3, \gamma_2]\gamma_3]\gamma_2] + 2[\Lambda_3, \gamma_3]\gamma_2]\gamma_2]$ modulo decomposables.

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